

COQUASI-BIALGEBRAS WITH PREANTIPODE AND RIGID MONOIDAL CATEGORIES.

PAOLO SARACCO

ABSTRACT. By a theorem of Majid, every monoidal category with a neutral tensor functor to finite-dimensional vector spaces gives rise to a coquasi-bialgebra. We prove that if the category is also rigid, then the associated coquasi-bialgebra admits a preantipode, providing in this way an analogue for coquasi-bialgebras of the well-known Ulbrich's reconstruction theorem for Hopf algebras.

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INTRODUCTION

A well-known result in the theory of Hopf algebras states that one can reconstruct, in a suitable way, a Hopf algebra from its category of finite-dimensional corepresentations. In details, if \mathcal{C} is a \mathbb{k} -linear, abelian, rigid symmetric monoidal category which is essentially small, and if $\mathcal{F} : \mathcal{C} \rightarrow \mathfrak{M}_f$ is an \mathbb{k} -linear, exact, faithful, monoidal functor, then there exists a commutative Hopf algebra H , unique up to isomorphism, such that \mathcal{F} factorizes through an equivalence of categories $\mathcal{F}^H : \mathcal{C} \rightarrow \mathfrak{M}_f^H$ followed by the obvious forgetful functor; in fact H represents the

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functor

$$R \rightarrow \text{Aut}^{\otimes}(\mathcal{F} \otimes R)$$

which associates to any commutative \mathbb{k} -algebra R the group of monoidal natural automorphisms of $\mathcal{F} \otimes R : \mathcal{C} \rightarrow \text{Mod}_R$ sending X to $\mathcal{F}(X) \otimes R$, see [R, Ch. II, Theorem 4.1.1, page 152], [DM, Theorem 2.11] and [JS, §§8, 9]. In particular, if \mathcal{C} is already the category of finite-dimensional right comodules over a commutative Hopf algebra A , then one can show that $A \cong H$ as Hopf algebras. In [U], Ulbrich showed that even in case the symmetry condition is dropped, it is still possible to provide an associated Hopf algebra H , even if not necessarily commutative.

In [M2], Majid partially extended this result to the framework of coquasi-bialgebras (or dual quasi-bialgebras), proving that if \mathcal{C} is an essentially small monoidal category endowed with a functor $\mathcal{F} : \mathcal{C} \rightarrow \mathfrak{M}_f$ that respects the tensor product in a suitable way, but that is not necessarily monoidal, then there exists a coquasi-bialgebra H such that \mathcal{F} factorizes through a monoidal functor $\mathcal{F}^H : \mathcal{C} \rightarrow {}^H\mathfrak{M}_f$ followed by the obvious forgetful functor.

In [AP], Ardizzoni and Pavarin introduced the notion of a preantipode for a coquasi-bialgebra, which is analogue to the that of an antipode for a bialgebra, in order to prove a structure theorem for coquasi-Hopf bicomodules. Since the existence of a preantipode turned out to be equivalent to the aforementioned structure theorem, the notion of a coquasi-bialgebra with preantipode has proved to be a closer analogue to that of a Hopf algebra with respect to the coquasi-Hopf algebra one. Furthermore, it follows indirectly from [Sc4, Theorem 2.6] that the existence of a preantipode is also equivalent to the rigidity of the category of finite-dimensional left comodules over the coquasi-bialgebra, as it happens for the Hopf algebra case.

Inspired from these results, we are going to prove an analogue of Ulbrich's theorem in the framework of coquasi-bialgebras. In detail, after recalling the definitions of coquasi-bialgebra and preantipode and their properties, we are going to show in Section 2 that if a category \mathcal{C} as in Majid's paper [M2] is also right rigid, then there exists a preantipode for the coendomorphism coquasi-bialgebra $H = \text{coend}(\mathcal{F})$ of the functor $\mathcal{F} : \mathcal{C} \rightarrow \mathfrak{M}_f$. This will provide us for a direct link between preantipodes and rigidity as well as for a different proof of Schauenburg's result and it will strengthen the affinity between antipodes and preantipodes at the same time. Moreover, this will allow us to explicitly reconstruct a coquasi-bialgebra with preantipode from its category of finite-dimensional left comodules, in the spirit of the classical Tannaka-Krein duality.

As an application of our achievements, we are going to show how to recover properties such as the uniqueness of the preantipode and the fact that any morphism of coquasi-bialgebras automatically preserves preantipodes, whenever they exist, from their categorical counterparts. We will also show how it is possible to endow the finite dual colagebra of a quasi-bialgebra with preantipode with a structure of coquasi-bialgebra with preantipode.

1. COQUASI-BIALGEBRAS AND PREANTIPODES

1.1. Monoidal categories and the notion of a coquasi-bialgebra. We recall some well-known notions and results concerning monoidal categories, coquasi-bialgebras and coquasi-Hopf bicomodules. A *monoidal category* $(\mathcal{C}, \otimes, \mathbb{I}, \alpha, \ell, \varphi)$ is a category \mathcal{C} equipped with a functor $\otimes : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$, called *tensor product*, and with a distinguished object \mathbb{I} , called the *unit*, such that \otimes is associative up to a natural

isomorphism α , \mathbb{I} is a left and right unit for \otimes up to natural isomorphisms ℓ and \wp respectively, and all diagrams involving α , ℓ and \wp commute (see e.g. [K, Chapter XI]). Formally, this means that we have three natural isomorphisms:

$$\begin{aligned}\alpha &: \otimes (\otimes \times \text{Id}_{\mathcal{C}}) \rightarrow \otimes (\text{Id}_{\mathcal{C}} \times \otimes) \quad (\text{associativity constraint}) \\ \ell &: \otimes (\mathbb{I} \times \text{Id}_{\mathcal{C}}) \rightarrow \text{Id}_{\mathcal{C}} \quad (\text{left unit constraint}) \\ \wp &: \otimes (\text{Id}_{\mathcal{C}} \times \mathbb{I}) \rightarrow \text{Id}_{\mathcal{C}} \quad (\text{right unit constraint})\end{aligned}$$

that satisfy the *Pentagon Axiom* and the *Triangle Axiom*, i.e. that make the following diagrams commutative

$$\begin{array}{ccc} & (M \otimes N) \otimes (P \otimes Q) & \\ \nearrow \alpha_{M \otimes N, P, Q} & \searrow \alpha_{M, N, P \otimes Q} & \\ ((M \otimes N) \otimes P) \otimes Q & & M \otimes (N \otimes (P \otimes Q)) \\ \downarrow \alpha_{M, N, P \otimes Q} & & \uparrow M \otimes \alpha_{N, P, Q} \\ (M \otimes (N \otimes P)) \otimes Q & & M \otimes ((N \otimes P) \otimes Q) \\ \nwarrow \alpha_{M, N \otimes P, Q} & & \end{array} \quad \begin{array}{ccc} & (M \otimes \mathbb{I}) \otimes N & \\ \nearrow \wp_{M \otimes \mathbb{I}, N} & \searrow \wp_{M \otimes \mathbb{I}, N} & \\ (M \otimes \mathbb{I}) \otimes N & & M \otimes N \\ \downarrow \alpha_{M, \mathbb{I}, N} & & \uparrow M \otimes \ell_N \\ M \otimes (\mathbb{I} \otimes N) & & \end{array}$$

for all M, N, P, Q in \mathcal{C} . A key example of a monoidal category is the category of vector spaces $(\mathfrak{M}, \otimes, \mathbb{k}, a, l, r)$ where a, l and r are the obvious isomorphisms. As in [Sc2, Definition 2.3.1], a *tensor functor* \mathcal{F} between monoidal categories $(\mathcal{C}, \otimes, \mathbb{I}, \alpha, \ell, \wp)$ and $(\mathcal{C}', \otimes', \mathbb{I}', \alpha', \ell', \wp')$ is a functor $\mathcal{F} : \mathcal{C} \rightarrow \mathcal{C}'$ together with an isomorphism $\varphi_0 : \mathbb{I}' \rightarrow \mathcal{F}(\mathbb{I})$ and a family of natural isomorphisms $\varphi_{X,Y} : \mathcal{F}(X) \otimes' \mathcal{F}(Y) \rightarrow \mathcal{F}(X \otimes Y)$ in \mathcal{C}' , for X, Y objects in \mathcal{C} . A tensor functor \mathcal{F} is said to be *neutral* if the following diagrams are commutative

$$(1) \quad \begin{array}{ccc} \mathbb{I}' \otimes \mathcal{F}(X) & \xrightarrow{\ell'_{\mathcal{F}(X)}} & \mathcal{F}(X) \\ \varphi_0 \otimes \mathcal{F}(X) \downarrow & & \uparrow \mathcal{F}(\ell_X) \\ \mathcal{F}(\mathbb{I}) \otimes \mathcal{F}(X) & \xrightarrow{\varphi_{\mathbb{I}, X}} & \mathcal{F}(\mathbb{I} \otimes' X) \end{array} \quad \begin{array}{ccc} \mathcal{F}(X) \otimes \mathbb{I}' & \xrightarrow{\wp'_{\mathcal{F}(X)}} & \mathcal{F}(X) \\ \mathcal{F}(X) \otimes \varphi_0 \downarrow & & \uparrow \mathcal{F}(\wp_X) \\ \mathcal{F}(X) \otimes \mathcal{F}(\mathbb{I}) & \xrightarrow{\varphi_{X, \mathbb{I}}} & \mathcal{F}(X \otimes' \mathbb{I}) \end{array}$$

Furthermore, \mathcal{F} is said to be *monoidal*¹ if (omitting the subscripts)

$$(2) \quad \begin{array}{ccc} (\mathcal{F}(X) \otimes' \mathcal{F}(Y)) \otimes' \mathcal{F}(Z) & \xrightarrow{\varphi(\varphi \otimes \mathcal{F}(Z))} & \mathcal{F}((X \otimes Y) \otimes Z) \\ \alpha' \downarrow & & \downarrow \mathcal{F}(\alpha) \\ \mathcal{F}(X) \otimes' (\mathcal{F}(Y) \otimes' \mathcal{F}(Z)) & \xrightarrow{\varphi(\mathcal{F}(X) \otimes \varphi)} & \mathcal{F}(X \otimes (Y \otimes Z)) \end{array}$$

commutes as well. It is said to be *strict* if φ_0 and φ are the identities. Since all the functors we are interested in are neutral, we will not specify it anymore.

The notions of (co)algebra and (co)module over a (co)algebra can be introduced in the general setting of monoidal categories (see e.g. [AM, §1.2], where (co)algebras are called (co)monoids). Given an algebra A in \mathcal{C} one can define the categories ${}_A\mathcal{C}$,

¹In [AM, Definition 3.5], these are called *strong* monoidal functors.

\mathcal{C}_A and ${}_A\mathcal{C}$ of left, right and two-sided modules over A respectively. Similarly, given a coalgebra C in \mathcal{C} , one can define the categories of C -comodules ${}^C\mathcal{C}, {}^C\mathcal{C}^C, {}^C\mathcal{C}^C$.

Henceforth and unless stated otherwise, we will fix a base field \mathbb{k} and we will assume to work in the monoidal category \mathfrak{M} of \mathbb{k} -vector spaces: all (co)algebras will be \mathbb{k} -(co)algebras, the unadorned tensor product \otimes will denote the tensor product over \mathbb{k} and $\text{Hom}(V, W)$ will be the set of \mathbb{k} -linear morphisms from V to W . The dimension of a vector space V will be denoted by d_V for shortness. In order to deal with comultiplications and coactions, we will also make use of the following variation of *Sweedler's Sigma Notation* (cf. [Sw, §1.2])

$$\Delta(x) := \sum x_1 \otimes x_2, \quad \rho_V^r(v) := \sum v_0 \otimes v_1, \quad \rho_W^l(w) := \sum w_{-1} \otimes w_0$$

for every coalgebra C , right C -comodule V , left C -comodule W and for all $x \in C$, $v \in V$ and $w \in W$. As a matter of notation, we will often omit the composition symbols between maps. Recall that if W is a left C -comodule, then its linear dual $W^* := \text{Hom}(W, \mathbb{k})$ is naturally a right C -comodule with $\sum f_0 \otimes f_1$ uniquely determined for all $w \in W$ by

$$(3) \quad \sum f_0(w) f_1 = \sum w_{-1} f(w_0).$$

Definition 1.1.1 (see e.g. [M2, §1]). A *coquasi-bialgebra* $(H, m, u, \Delta, \varepsilon, \omega)$ is a coassociative and counital coalgebra (H, Δ, ε) endowed with a multiplication $m : H \otimes H \rightarrow H$ and a unit $u : \mathbb{k} \rightarrow H$, which are coalgebra morphisms, and with a convolution invertible linear map $\omega : H \otimes H \otimes H \rightarrow \mathbb{k}$ such that

$$(4) \quad \omega(H \otimes H \otimes m) * \omega(m \otimes H \otimes H) = (\varepsilon \otimes \omega) * \omega(H \otimes m \otimes H) * (\omega \otimes \varepsilon),$$

$$(5) \quad \omega(h \otimes 1_H \otimes k) = \varepsilon(h) \varepsilon(k)$$

$$(6) \quad m(H \otimes m) * \omega = \omega * m(m \otimes H),$$

$$(7) \quad m(1_H \otimes h) = h, \quad m(h \otimes 1_H) = h,$$

for all $h, k \in H$, where $*$ denotes the convolution product and $1_H := u(1_{\mathbb{k}})$. We say that m is *quasi-associative* and we refer to ω as the *reassociator* of the coquasi-bialgebra. A *morphism of coquasi-bialgebras*

$$f : (H, m, u, \Delta, \varepsilon, \omega) \rightarrow (H', m', u', \Delta', \varepsilon', \omega')$$

is a coalgebra homomorphism $f : (H, \Delta, \varepsilon) \rightarrow (H', \Delta', \varepsilon')$ such that

$$m'(f \otimes f) = f m, \quad f u = u', \quad \omega'(f \otimes f \otimes f) = \omega.$$

It is an isomorphism if, in addition, it is invertible.

Remark 1.1.2. As a matter of simplicity, we will often leave the structure maps out when referring to coquasi-bialgebras, i.e. we will simply say that H is a coquasi-bialgebra, without explicitly mentioning $\Delta, \varepsilon, m, u$ and ω .

Dually to coquasi-bialgebras we have *quasi-bialgebras*, i.e. ordinary algebras A with a counital comultiplication which is coassociative up to conjugation by a suitable invertible element $\Phi \in A \otimes A \otimes A$. Notice that the definition of a quasi-bialgebra precedes that of a coquasi-bialgebra and dates back to [D, §1].

Let H be a coquasi-bialgebra. The category ${}^H\mathfrak{M}$ of left H -comodules becomes a monoidal category in such a way that the underlying functor ${}^H\mathcal{U} : {}^H\mathfrak{M} \rightarrow \mathfrak{M}$ is a strict tensor functor. Explicitly, given two left H -comodules V and W , their tensor product is an H -comodule via the diagonal coaction, i.e. $\rho_{V \otimes W}^l(v \otimes w) =$

$\sum v_{-1}w_{-1} \otimes v_0 \otimes w_0$. The unit is \mathbb{k} , regarded as a left H -comodule via the trivial coaction $\rho_{\mathbb{k}}^l(k) = 1_H \otimes k$. Associativity and unit constraints are given by

$$(8) \quad {}^H\alpha_{U,V,W}(u \otimes v \otimes w) := \sum \omega^{-1}(u_{-1} \otimes v_{-1} \otimes w_{-1})u_0 \otimes v_0 \otimes w_0, \\ l_U(k \otimes u) := ku \quad \text{and} \quad r_U(u \otimes k) := uk,$$

for every $U, V, W \in {}^H\mathfrak{M}$ and all $u \in U, v \in V, w \in W, k \in \mathbb{k}$. The monoidal category we have just described will be denoted by $({}^H\mathfrak{M}, \otimes, \mathbb{k}, {}^H\alpha, l, r)$. Notice that every morphism of coquasi-bialgebras $f : H \rightarrow H'$ induces a strict monoidal functor $f\mathfrak{M} : {}^H\mathfrak{M} \rightarrow {}^{H'}\mathfrak{M}$, which is given by the assignments

$$f\mathfrak{M}(X, \rho_X^H : X \rightarrow H \otimes X) = (X, (f \otimes X) \rho_X^H), \quad f\mathfrak{M}(\gamma : X \rightarrow Y) = \gamma.$$

Similarly, the monoidal categories $(\mathfrak{M}^H, \otimes, \mathbb{k}, \alpha^H, l, r)$ and $({}^H\mathfrak{M}^H, \otimes, \mathbb{k}, {}^H\alpha^H, l, r)$ are introduced. We just point out that

$$(9) \quad \alpha_{U,V,W}^H((u \otimes v) \otimes w) := \sum u_0 \otimes (v_0 \otimes w_0)\omega(u_1 \otimes v_1 \otimes w_1), \\ {}^H\alpha_{U,V,W}^H(u \otimes v \otimes w) := \sum \omega^{-1}(u_{-1} \otimes v_{-1} \otimes w_{-1})u_0 \otimes v_0 \otimes w_0\omega(u_1 \otimes v_1 \otimes w_1).$$

Remark 1.1.3. Given a coquasi-bialgebra H , we cannot construct the category \mathfrak{M}_H of right H -modules in general, as H is not an algebra in \mathfrak{M} . The same happens if we consider H as an object in \mathfrak{M}^H or in ${}^H\mathfrak{M}$. On the other hand, H as a bicomodule over itself with $\rho_H^l = \rho_H^r = \Delta$ turns out to be an algebra in the monoidal category $({}^H\mathfrak{M}^H, \otimes, \mathbb{k}, {}^H\alpha^H, l, r)$ with its multiplication and unit. Thus we may consider the right H -modules in the category ${}^H\mathfrak{M}^H$:

$${}^H\mathfrak{M}_H^H := ({}^H\mathfrak{M}^H)_H$$

and what we get is the so-called category of *right coquasi-Hopf H -bicomodules* [BC, Remark 2.3]. Notice that we may perform ${}^H_H\mathfrak{M}^H$ as well and this is the category of *left coquasi-Hopf H -bicomodules*.

1.2. Preantipodes for coquasi-bialgebras. Define functors $F : {}^H\mathfrak{M} \rightarrow {}^H\mathfrak{M}_H^H$ and $G : {}^H\mathfrak{M}_H^H \rightarrow {}^H\mathfrak{M}$ by $F(V) := V \otimes H$ and $G(M) := M^{\text{co}H}$ respectively, where $M^{\text{co}H} := \{m \in M \mid m_0 \otimes m_1 = m \otimes 1_H\}$ is the space of right H -coinvariant elements in M . The structures on $F(V)$ are given explicitly by

$$\rho_{V \otimes H}^l(v \otimes h) = \sum v_{-1}h_1 \otimes (v_0 \otimes h_2), \quad \rho_{V \otimes H}^r(v \otimes h) = \sum (v \otimes h_1) \otimes h_2, \\ \mu_{V \otimes H}^r((v \otimes h) \otimes l) = (v \otimes h)l = \sum \omega^{-1}(v_{-1} \otimes h_1 \otimes l_1)v_0 \otimes h_2l_2,$$

for every $v \in V$ and $h, l \in H$. The functor F turns out to be left adjoint to the functor G , where the counit $\epsilon : FG \rightarrow \text{id}$ and the unit $\eta : \text{id} \rightarrow GF$ of the adjunction are given respectively by

$$(10) \quad \epsilon_M(x \otimes h) := xh \quad \text{and} \quad \eta_N(n) := n \otimes 1_H,$$

for every $M \in {}^H\mathfrak{M}_H^H$, $N \in {}^H\mathfrak{M}$ and for all $m \in M, n \in N, h \in H$. Moreover η is a natural isomorphism, whence the functor F is full and faithful (cf. e.g. the right-hand version of [Sc4, Lemma 2.1]).

Definition 1.2.1 ([AP, Definition 3.6]). A preantipode for a coquasi-bialgebra H is a \mathbb{k} -linear map $S : H \rightarrow H$ such that, for all $h \in H$,

$$(11) \quad \sum S(h_1)_1 h_2 \otimes S(h_1)_2 = 1_H \otimes S(h),$$

$$(12) \quad \sum S(h_2)_1 \otimes h_1 S(h_2)_2 = S(h) \otimes 1_H,$$

$$(13) \quad \sum \omega(h_1 \otimes S(h_2) \otimes h_3) = \varepsilon(h).$$

Remark 1.2.2. Let H be a coquasi-bialgebra with a preantipode S . Then

$$(14) \quad \sum h_1 S(h_2) = \varepsilon S(h) 1_H = \sum S(h_1) h_2$$

for all $h \in H$. In particular, if $\varepsilon S(h) = \varepsilon(h)$ then S is an ordinary antipode.

Theorem 1.2.3 (Structure Theorem for coquasi-Hopf bicomodules, [AP, Theorem 3.9]). *Let H be a coquasi-bialgebra. The following assertions are equivalent:*

- (i) *The adjunction (F, G) is an equivalence of categories.*
- (ii) *There exists a preantipode.*

As for the quasi-bialgebra case (cf. [Sa, Theorem 5]), the preantipode for a coquasi-bialgebra has to be unique, whenever it exists.

Theorem 1.2.4. *If a coquasi-bialgebra H admits a preantipode, then it is unique.*

Proof. Recall from the proof of [AP, Theorem 3.5] that if a preantipode S exists then an inverse for the component of the counit $\epsilon_{H \hat{\otimes} H} : (H \hat{\otimes} H)^{\text{co}H} \otimes H \rightarrow H \hat{\otimes} H$ is explicitly given by the relation

$$(15) \quad \epsilon_{H \hat{\otimes} H}^{-1}(h \otimes 1) = \sum h_1 \otimes S(h_2) \otimes h_3.$$

If S and T are two preantipodes for H then from (15) we get

$$\sum h_1 \otimes S(h_2) \otimes h_3 = \sum h_1 \otimes T(h_2) \otimes h_3$$

for every $h \in H$. By applying $\varepsilon \otimes H \otimes \varepsilon$ to both sides of this identity we get that $S(h) = T(h)$ for every $h \in H$, proving the claim. \square

2. COQUASI-BIALGEBRAS WITH PREANTIPODE AND RIGID MONOIDAL CATEGORIES

It is already clear from the left-handed version of [Sc4, Theorem 2.6] and from [AP, Theorem 3.9] that the existence of a preantipode for a coquasi-bialgebra is strictly related with the category of its finite-dimensional comodules being rigid. However, we have no explicit link between these two properties, differently from what happens in the Hopf algebra case. Our aim in this section is to relate directly the existence of the preantipode for a coquasi-bialgebra H with the existence of dual objects for every comodule in ${}^H\mathfrak{M}_f$ and, in the spirit of Tannaka-Krein reconstruction theory, to show how it is possible to recover explicitly the first from the seconds.

2.1. The classical results. The main reference for this subsection will be [Sc3]. All the results are supposed to be well-known, but we are going to retrieve the main steps of the classical reconstruction process for the sake of completeness. Occasionally, we are going to give further references to connect the present material with the wide literature on the subject.

Initially, let \mathcal{C} be an essentially small category (i.e. a category which is equivalent to a small one) equipped with a functor $\mathcal{F} : \mathcal{C} \rightarrow \mathfrak{M}_f$ from \mathcal{C} into the category of finite-dimensional vector spaces. Such a functor is sometimes denoted by ω , as it happens for example in [R, Sc3, U], but we decided not to adopt this notation

in order to avoid confusion with the reassociator for a coquasi-bialgebra. For every vector space V , denote by $\text{Nat}(\mathcal{F}, V \otimes \mathcal{F})$ the set of natural transformations between \mathcal{F} and the functor $V \otimes \mathcal{F} : \mathcal{C} \rightarrow \mathfrak{M}$ mapping every object X in \mathcal{C} to the object $V \otimes \mathcal{F}(X)$ in \mathfrak{M} . Then the functor

$$\text{Nat}(\mathcal{F}, - \otimes \mathcal{F}) : \mathfrak{M} \rightarrow \underline{\text{Set}}$$

is represented by a coalgebra $H_{\mathcal{F}}$, also denoted by $\underline{\text{coend}}(\mathcal{F})$ or by $\int^X \mathcal{F}(X)^* \otimes \mathcal{F}(X)$, which is called the *coendomorphism coalgebra of \mathcal{F}* (or of $(\mathcal{C}, \mathcal{F})$ if some confusion may arise). That means that we have a natural isomorphism

$$(16) \quad \vartheta : \text{Hom}(H_{\mathcal{F}}, -) \cong \text{Nat}(\mathcal{F}, - \otimes \mathcal{F}).$$

Since \mathcal{F} is fixed, we may write H instead of $H_{\mathcal{F}}$. As a vector space, H is defined to be the coend of the functor $\mathcal{F} \otimes \mathcal{F}^* = \otimes(\mathcal{F} \times \mathcal{F}^*)$ from $\mathcal{C} \times \mathcal{C}^{\text{op}}$ to \mathfrak{M} (see e.g. [ML, §IX.6] for details about the coend construction). It is endowed with a coalgebra structure as follows. Consider the natural transformation $\rho^H := \vartheta_H(\text{id}_H) : \mathcal{F} \rightarrow H \otimes \mathcal{F}$, which we will denote by ρ when no confusion may arise. The comultiplication Δ and the counit ε are the unique linear maps such that $\vartheta_{H \otimes H}(\Delta) = (H \otimes \rho) \rho$ and $\vartheta_{\mathbb{k}}(\varepsilon) = \text{id}_{\mathcal{F}}$. Naturality of ϑ implies that for all V in \mathfrak{M} and $f \in \text{Hom}(H, V)$

$$(17) \quad \vartheta_V(f) = (\vartheta_V \text{Hom}(H, f))(\text{id}_H) = (\text{Nat}(\mathcal{F}, f \otimes \mathcal{F}) \vartheta_H)(\text{id}_H) = (f \otimes \mathcal{F}) \rho.$$

Therefore, $\vartheta_V(f)_X = (f \otimes \mathcal{F}(X)) \rho_X$ for every X in \mathcal{C} . Furthermore, from

$$(18a) \quad (\Delta \otimes \mathcal{F}(X)) \rho_X = \vartheta_{H \otimes H}(\Delta)_X = (H \otimes \rho_X) \rho_X$$

$$(18b) \quad \text{and } (\varepsilon \otimes \mathcal{F}(X)) \rho_X = \vartheta_{\mathbb{k}}(\varepsilon)_X = \text{id}_{\mathcal{F}(X)}$$

we deduce that every vector space $\mathcal{F}(X)$ is automatically endowed with an H -comodule structure $(\mathcal{F}(X), \rho_X)$. Thus \mathcal{F} factorizes through the obvious forgetful functor $\mathcal{U} : {}^H\mathfrak{M} \rightarrow \mathfrak{M}$, i.e. we have $\mathcal{F}^H : \mathcal{C} \rightarrow {}^H\mathfrak{M}$ such that $\mathcal{U}\mathcal{F}^H = \mathcal{F}$. Moreover, H enjoys the following universal property: if C is another coalgebra and if $\mathcal{G} : \mathcal{C} \rightarrow {}^C\mathfrak{M}$ is another functor such that $\mathcal{U}\mathcal{G} = \mathcal{F}$, then there is a unique morphism of coalgebras $\epsilon : H \rightarrow C$, given as the image of the C -coaction $\rho^C \in \text{Nat}(\mathcal{F}, C \otimes \mathcal{F})$ in $\text{Hom}(H, C)$ via (16), that induces a functor ${}^{\epsilon}\mathfrak{M} : {}^H\mathfrak{M} \rightarrow {}^C\mathfrak{M}$ such that ${}^{\epsilon}\mathfrak{M}\mathcal{F}^H = \mathcal{G}$.

Remark 2.1.1. Observe that any property that we assign to $H = \underline{\text{coend}}(\mathcal{F})$, in fact it holds for every representing object R in \mathfrak{M} of the functor $\text{Nat}(\mathcal{F}, - \otimes \mathcal{F})$ and we will eventually take advantage of this in what follows. For example, if \mathcal{C} is already the category ${}^C\mathfrak{M}_f$ of finite-dimensional comodules over a coalgebra C and if \mathcal{F} is already the forgetful functor \mathcal{U} , then we may choose C as a representing object for $\text{Nat}(\mathcal{U}, - \otimes \mathcal{U})$ (cf. e.g. [Sc3, Lemma 2.2.1]). In this case, the comultiplication and the counit of C already satisfy relations (18), whence they are the unique morphisms Δ and ε induced on the vector space C by the isomorphism (16) as above.

Assume further that $(\mathcal{C}, \boxtimes, \mathbb{I}, \alpha, \ell, \wp)$ is monoidal and that \mathcal{F} is a tensor functor. This means that in \mathfrak{M} we have a family of isomorphisms $\varphi_{X,Y} : \mathcal{F}(X) \otimes \mathcal{F}(Y) \rightarrow \mathcal{F}(X \boxtimes Y)$ which is natural in both components and an isomorphism $\varphi_0 : \mathbb{k} \rightarrow \mathcal{F}(\mathbb{I})$ compatible with the left and right unit constraints, where the compatibility is expressed by the commutativity of the diagrams (1). Following [M2], H can be endowed with a coquasi-bialgebra structure as follows.

Remark 2.1.2. We point out that here the associativity constraint satisfies $\alpha : \boxtimes(\boxtimes \times -) \rightarrow \boxtimes(- \times \boxtimes)$ (see §1.1), while in [M2] it goes the other way around.

Furthermore, in [M2] there's no explicit reference to the unitality of the multiplication. Nevertheless, we will see that this does not create any difficulty.

To show where the multiplication and the reassociator come from, notice that also the functors $\text{Nat}(\mathcal{F}^n, - \otimes \mathcal{F}^n)$ for $n \geq 2$ are representable, where $\mathcal{F}^n : \mathcal{C}^n \rightarrow \mathfrak{M}_f$ maps every n -uple of objects (X_1, \dots, X_n) to the tensor product $\mathcal{F}(X_1) \otimes \dots \otimes \mathcal{F}(X_n)$. They are represented by the n -fold tensor product $H^{\otimes n}$, i.e. we have natural isomorphisms $\vartheta_V^n : \text{Hom}(H^{\otimes n}, V) \cong \text{Nat}(\mathcal{F}^n, V \otimes \mathcal{F}^n)$ for all $n \geq 1$ and for all V . As a matter of notation and if no confusion may arise, given an arrow $f : X \rightarrow Y$ and other two objects V and W in a category \mathcal{C} with tensor product \boxtimes , we will eventually write simply f instead of $V \boxtimes f \boxtimes W$, leaving the identity morphisms out and understanding that f has to be applied to the suitable tensorand. With this convention we have explicitly

$$\vartheta_V^n(f)_{X_1 \dots X_n} = (f \otimes \mathcal{F}^n(X_1, \dots, X_n)) \tau_{\mathcal{F}^{n-1}(X_1, \dots, X_{n-1}), H} \cdot \dots \tau_{\mathcal{F}(X_1), H} (\rho_{X_1} \otimes \dots \otimes \rho_{X_n})$$

where for every pair of objects V, W in \mathfrak{M} , $\tau_{V, W} : V \otimes W \rightarrow W \otimes V$ denotes the natural transformation acting as $\tau_{V, W}(v \otimes w) = w \otimes v$ (see the sentence above [M2, Lemma 2.3]; cf. also [Sc3, Lemma 2.3.6]).

As a consequence of the representability of the functors $\text{Nat}(\mathcal{F}^n, - \otimes \mathcal{F}^n)$, we can define the multiplication $m : H \otimes H \rightarrow H$ as the unique map such that

$$(19) \quad (H \otimes \varphi_{X, Y}) \vartheta_H^2(m)_{X, Y} = \rho_{X \boxtimes Y} \varphi_{X, Y}$$

for all X, Y in \mathcal{C} . It is a coalgebra morphism (see [M2, Lemma 2.4]). In turn, the reassociator $\omega \in (H \otimes H \otimes H)^*$ is the unique such that for all X, Y, Z in \mathcal{C}

$$(20) \quad \varphi_{X \boxtimes Y, Z} (\varphi_{X, Y} \otimes \mathcal{F}(Z)) \vartheta_{\mathbb{k}}^3(\omega)_{X, Y, Z} = F(\alpha_{X, Y, Z}^{-1}) \varphi_{X, Y \boxtimes Z} (\mathcal{F}(X) \otimes \varphi_{Y, Z}).$$

Note that the case $n = 0$, which will give rise to the unit, is not considered above and in fact should be treated separately. Set, by definition, $\mathcal{F}^0(X) \equiv \mathbb{k}$ for all X in \mathcal{C} . It follows that $\text{Nat}(\mathcal{F}^0, V \otimes \mathcal{F}^0)$ is just $\text{Hom}(\mathbb{k}, V \otimes \mathbb{k})$ for every vector space V . We have a natural isomorphism $\vartheta^0 : \text{Hom}(\mathbb{k}, -) \cong \text{Nat}(\mathcal{F}^0, - \otimes \mathcal{F}^0)$ as well, which is given by $\vartheta_V^0(f)_X = r_V^{-1} f$ with inverse $(\vartheta_V^0)^{-1}(\eta) = r_V \eta_{\mathbb{I}}$ for all X in \mathcal{C} , V in \mathfrak{M} , $f \in \text{Hom}(\mathbb{k}, V)$ and $\eta \in \text{Nat}(\mathcal{F}^0, V \otimes \mathcal{F}^0)$. The unit on H is then the unique morphism $u : \mathbb{k} \rightarrow H$ such that

$$(21) \quad (H \otimes \varphi_0) \vartheta_H^0(u) = \rho_{\mathbb{I}} \varphi_0.$$

Explicitly, it is given by the composition

$$(22) \quad \mathbb{k} \xrightarrow{\varphi_0} \mathcal{F}(\mathbb{I}) \xrightarrow{\rho_{\mathbb{I}}} H \otimes \mathcal{F}(\mathbb{I}) \xrightarrow{H \otimes \varphi_0^{-1}} H \otimes \mathbb{k} \xrightarrow{r_H} H.$$

Observe that, since unitality is unrelated with the coherence condition (2), the unit is the same that has been constructed from ordinary monoidal functors in [U, page 255] and in [Sc3, Corollary 2.3.7], for example.

Remark 2.1.3. If C is a coalgebra endowed with two morphisms of coalgebras $\mu : C \otimes C \rightarrow C$ and $\eta : \mathbb{k} \rightarrow C$, then the tensor product $M \otimes N$ of two left C -comodules (M, ρ_M) , (N, ρ_N) is still a C -comodule with $\text{coact}_{M \otimes N} = (\mu \otimes M \otimes N) \tau_{M, H}(\rho_M \otimes \rho_N)$, and \mathbb{k} is a C -comodule with $\text{coact}_{\mathbb{k}} = r_C^{-1} \eta$. By definition of m , for every X, Y in \mathcal{C} we have

$$\vartheta_H^2(m)_{X, Y} = (m \otimes \mathcal{F}(X) \otimes \mathcal{F}(Y)) \tau_{\mathcal{F}(X), H} (\rho_X \otimes \rho_Y) = \text{coact}_{\mathcal{F}(X) \otimes \mathcal{F}(Y)}.$$

It follows from (19) that $\rho_{X \boxtimes Y} \varphi_{X,Y} = (H \otimes \varphi_{X,Y}) \text{coact}_{\mathcal{F}(X) \otimes \mathcal{F}(Y)}$ and all the morphisms $\varphi_{X,Y} : \mathcal{F}(X) \otimes \mathcal{F}(Y) \rightarrow \mathcal{F}(X \boxtimes Y)$ are comodule morphisms. Moreover, $\vartheta_H^0(u) = \text{coact}_{\mathbb{k}}$ and from (22) it follows that φ_0 is H -colinear as well, as

$$(23) \quad \rho_1 \varphi_0 = (H \otimes \varphi_0) \text{coact}_{\mathbb{k}}.$$

Thus, we see that the comodule structures on the tensor product $X \boxtimes Y$ and on the unit object \mathbb{I} are compatible with the monoidal structure defined on the category ${}^H\mathfrak{M}$, in the sense that the functor $\mathcal{F}^H : \mathcal{C} \rightarrow {}^H\mathfrak{M}$ is a tensor functor. Furthermore, by definition of ω , for all $x \in \mathcal{F}(X)$, $y \in \mathcal{F}(Y)$ and $z \in \mathcal{F}(Z)$,

$$\vartheta_{\mathbb{k}}^3(\omega)_{X,Y,Z}(x \otimes (y \otimes z)) = \sum \omega(x_{-1} \otimes y_{-1} \otimes z_{-1})(x_0 \otimes y_0) \otimes z_0.$$

This is exactly the (inverse of the) associativity constraint in the category of left H -comodules (cf. relation (8)). Relation (20) encodes the fact that the functor $\mathcal{F}^H : \mathcal{C} \rightarrow {}^H\mathfrak{M}$ is in fact a monoidal functor, differently from $\mathcal{F} : \mathcal{C} \rightarrow \mathfrak{M}$.

Lemma 2.1.4. *The reassociator ω and the multiplication m are unital, in the sense that conditions (5) and (7) are satisfied.*

Proof. Since the detailed computations would be too technical and out of the purposes of this paper, let us give only a sketch of the proof. Denote temporarily by $H^{(i)}$ the i -th tensorand in $H^{\otimes n}$ for some $n \geq 1$ and some $1 \leq i \leq n$. By omitting (some of) the identity arrows, the following computation

$$\begin{aligned} & (H \otimes H \otimes l_{\mathcal{F}(X)})(H \otimes H \otimes \varphi_0^{-1} \otimes \mathcal{F}(X)) \tau_{\mathcal{F}(\mathbb{I}),H}(\rho_1 \otimes \rho_X)(\varphi_0 \otimes \mathcal{F}(X)) l_{\mathcal{F}(X)}^{-1} \\ &= (H \otimes r_H \otimes \mathcal{F}(X)) \tau_{\mathbb{k},H}(H \otimes \varphi_0^{-1} \otimes H \otimes \mathcal{F}(X))(\rho_1 \otimes \rho_X)(\varphi_0 \otimes \mathcal{F}(X)) l_{\mathcal{F}(X)}^{-1} \\ &= (H \otimes l_H \otimes \mathcal{F}(X))(H \otimes \varphi_0^{-1} \otimes H \otimes \mathcal{F}(X))(\rho_1 \otimes \rho_X)(\varphi_0 \otimes \mathcal{F}(X)) l_{\mathcal{F}(X)}^{-1} \\ &= (u \otimes \rho_X) l_{\mathcal{F}(X)}^{-1} = (u l_H^{-1} \otimes \mathcal{F}(X)) \rho_X, \end{aligned}$$

where the last equality follows from $(\mathbb{k} \otimes \rho_X) l_{\mathcal{F}(X)}^{-1} = l_{H \otimes \mathcal{F}(X)}^{-1} \rho_X = (l_H^{-1} \otimes \mathcal{F}(X)) \rho_X$, shows that we have

$$u_{H^{(i)}} l_{H^{(i)}}^{-1} \tau(\rho_{X_1} \otimes \cdots \otimes \rho_{X_n}) = l_{\mathcal{F}(X_i)} \varphi_0^{-1} \tau'(\rho_{X_1} \otimes \cdots \otimes \rho_{X_i} \otimes \cdots \otimes \rho_{X_n}) \varphi_0 l_{\mathcal{F}(X_i)}^{-1}$$

where

$$\begin{aligned} \tau &= \tau_{\mathcal{F}^{n-1}(X_1, \dots, X_{n-1}),H} \cdots \tau_{\mathcal{F}(X_1),H} \quad \text{and} \\ \tau' &= \tau_{\mathcal{F}^n(X_1, \dots, \mathbb{I}, \dots, X_{n-1}),H} \cdots \tau_{\mathcal{F}^i(X_1, \dots, X_{i-1}, \mathbb{I}),H} \cdots \tau_{\mathcal{F}(X_1),H}. \end{aligned}$$

Moreover, in view of relation (1) we have that $(\varphi_0 \otimes \mathcal{F}(X)) l_{\mathcal{F}(X)}^{-1} = \varphi_{1,X}^{-1} \mathcal{F}(l_X^{-1})$. Therefore, if $f : H^{\otimes n+1} \rightarrow V$ for V a vector space, then

$$\vartheta_V^n(f u_{H^{(i)}} l_{H^{(i)}}^{-1})_{X_1, \dots, X_n} \mathcal{F}(l_{X_i}) \varphi_{1,X_i} = \mathcal{F}(l_{X_i}) \varphi_{1,X_i} \vartheta_V^{n+1}(f)_{X_1, \dots, \mathbb{I}, \dots, X_n}$$

By using this relation and the naturality of $\vartheta_H(\text{id}_H)$, one checks that

$$\begin{aligned} \vartheta_H(m(u \otimes H) l_H^{-1})_X \mathcal{F}(l_X) &= (H \otimes \mathcal{F}(l_X)) (H \otimes \varphi_{1,X}) \vartheta_H^2(m)_{1,X} \varphi_{1,X}^{-1} \\ &\stackrel{(19)}{=} (H \otimes \mathcal{F}(l_X)) \vartheta_H(\text{id}_H)_{\mathbb{I} \boxtimes X} = \vartheta_H(\text{id}_H)_X \mathcal{F}(l_X) \end{aligned}$$

for all X in \mathcal{C} , which means that $m(u \otimes H) l_H^{-1} = \text{id}_H$ as desired. Unitality on the other side may be checked similarly, since formulas as the ones above hold for the

right unit constraint as well. Let us conclude with the unitality of ω . We already know that

$$\varphi_{X \boxtimes \mathbb{I}, Y} (\varphi_{X, \mathbb{I}} \otimes \mathcal{F}(Y)) \vartheta_{\mathbb{K}}^3(\omega)_{X, \mathbb{I}, Y} = \mathcal{F}(\alpha_{X, \mathbb{I}, Y}^{-1}) \varphi_{X, \mathbb{I} \boxtimes Y} (\mathcal{F}(X) \otimes \varphi_{\mathbb{I}, Y}).$$

As above, we may compute

$$\begin{aligned} & \vartheta_{\mathbb{K}}^2(\omega(H \otimes u \otimes H)(H \otimes l_H^{-1})) \\ & \stackrel{(*)}{=} (\mathcal{F}(\varphi_X) \otimes \mathcal{F}(Y)) (\varphi_{X, \mathbb{I}} \otimes \mathcal{F}(Y)) \vartheta_{\mathbb{K}}^3(\omega)_{X, \mathbb{I}, Y} (\mathcal{F}(X) \otimes \varphi_{\mathbb{I}, Y}^{-1}) (\mathcal{F}(X) \otimes \mathcal{F}(\ell_Y^{-1})) \\ & = (\mathcal{F}(\varphi_X) \otimes \mathcal{F}(Y)) \varphi_{X \boxtimes \mathbb{I}, Y}^{-1} \mathcal{F}(\alpha_{X, \mathbb{I}, Y}^{-1}) \varphi_{X, \mathbb{I} \boxtimes Y} (\mathcal{F}(X) \otimes \mathcal{F}(\ell_Y^{-1})) \\ & = \varphi_{X, Y}^{-1} \mathcal{F}(\varphi_X \boxtimes Y) \mathcal{F}(\alpha_{X, \mathbb{I}, Y}^{-1}) \mathcal{F}(X \boxtimes \ell_Y^{-1}) \varphi_{X, Y} \\ & = \text{id}_{\mathcal{F}(X) \otimes \mathcal{F}(Y)} = \vartheta_{\mathbb{K}}^2(m_{\mathbb{K}}(\varepsilon \otimes \varepsilon)), \end{aligned}$$

where in $(*)$ we implicitly used the fact that

$$(\mathcal{F}(X) \otimes \mathcal{F}(\ell_Y)) (\mathcal{F}(X) \otimes \varphi_{\mathbb{I}, Y}) = (\mathcal{F}(\varphi_X) \otimes \mathcal{F}(Y)) (\varphi_{X, \mathbb{I}} \otimes \mathcal{F}(Y)).$$

Thus, $\omega(x \otimes 1_H \otimes y) = \varepsilon(x) \varepsilon(y)$ for all $x, y \in H$. \square

Summing up, we have the following central result.

Theorem 2.1.5 (see [M2, Theorem 2.2]). *Let $(\mathcal{C}, \boxtimes, \mathbb{I}, \alpha, \ell, \varphi)$ be an essentially small monoidal category and let $(\mathcal{F}, \varphi, \varphi_0) : \mathcal{C} \rightarrow \mathfrak{M}_f$ be a tensor functor. Then there is a coquasi-bialgebra H , unique up to isomorphism, universal with the property that \mathcal{F} factorizes as a monoidal functor $\mathcal{F}^H : \mathcal{C} \rightarrow {}^H\mathfrak{M}$ followed by the forgetful functor. Universal means that if H' is another such coquasi-bialgebra then there is a unique map of coquasi-bialgebras $\epsilon : H \rightarrow H'$ inducing a functor ${}^\epsilon\mathfrak{M} : {}^H\mathfrak{M} \rightarrow {}^{H'}\mathfrak{M}$ such that the following commutes*

$$\begin{array}{ccc} & \mathcal{C} & \\ \mathcal{F}^H \swarrow & & \searrow \mathcal{F}^{H'} \\ {}^H\mathfrak{M} & \xrightarrow{f_{\mathfrak{M}}} & {}^{H'}\mathfrak{M} \end{array}$$

Remark 2.1.6. Let B be a coquasi-bialgebra. Consider the monoidal category of finite-dimensional left B -comodules ${}^B\mathfrak{M}_f$ together with the forgetful functor $\mathcal{U} : {}^B\mathfrak{M}_f \rightarrow \mathfrak{M}_f$, which is a tensor functor. As in Remark 2.1.1, we may choose B itself as a representing object for $\text{Nat}(\mathcal{U}, - \otimes \mathcal{U})$ and in this way we recover its comultiplication and counit. Moreover, since the original m , u and ω of B satisfy (19), (20) and (21) respectively, we recover these as well as the structure maps provided by Theorem 2.1.5.

2.2. General consequences of rigidity. The main objective of this paper is to show that if the category \mathcal{C} is also right rigid, i.e. if every object X in \mathcal{C} admits a right dual X^* still in \mathcal{C} , then the coquasi-bialgebra H representing $\text{Nat}(\mathcal{F}, - \otimes \mathcal{F})$ admits a preantipode. To this aim, let us recall some general consequences of the rigidity of \mathcal{C} .

Definition 2.2.1. A *right dual object* X^* of X in \mathcal{C} is a triple $(X^*, \text{ev}_X, \text{db}_X)$ in which X^* is an object in \mathcal{C} and

$$\text{ev}_X : X \boxtimes X^* \rightarrow \mathbb{I} \quad \text{and} \quad \text{db}_X : \mathbb{I} \rightarrow X^* \boxtimes X$$

are morphisms in \mathcal{C} , called *evaluation* and *dual basis*² respectively, that satisfies

$$(24) \quad (\text{ev}_X \boxtimes X) \alpha_{X, X^*, X}^{-1} (X \boxtimes \text{db}_X) = \text{id}_X,$$

$$(25) \quad (X^* \boxtimes \text{ev}_X) \alpha_{X^*, X, X^*} (\text{db}_X \boxtimes X^*) = \text{id}_{X^*}.$$

An object which admits a right dual object is said to be *right rigid* (or *dualizable*). If every object in \mathcal{C} is right rigid, then we say that \mathcal{C} is *right rigid*. If $f : X \rightarrow Y$ is a morphism between right rigid objects in a monoidal category \mathcal{C} then (omitting the unit constraints) its *dual map* (or *transpose*) is given by the composition

$$(26) \quad f^* := (X^* \boxtimes \text{ev}_Y) (X^* \boxtimes (f \boxtimes Y^*)) \alpha_{X^*, X, Y^*} (\text{db}_X \boxtimes Y^*).$$

Dually one defines *left duals* and *left rigid* monoidal categories.

We will often refer to right dual objects simply as *right duals* or just *duals*.

Example 2.2.2. *The classical example of dualizable objects is provided by finite-dimensional vector spaces. Given V in \mathfrak{M}_f with basis $\{v_i \mid i = 1, \dots, d_V\}$ there exists elements $v^i \in V^*$ for $1 \leq i \leq d_V$ such that $v^j(v_i) = \delta_{ij}$ (the Kronecker delta) for all $1 \leq i, j \leq d_V$. The evaluation map is simply $\text{ev}_V(v \otimes f) = f(v)$ for $v \in V$ and $f \in V^*$. The dual basis map is given by $\text{db}_V(1_k) = \sum_{i=1}^{d_V} v^i \otimes v_i$. For all $u \in V$, $f \in V^*$, relations (24) and (25) amounts to the well-known*

$$(27) \quad u = \sum_{i=1}^{d_V} v^i(u) v_i \quad \text{and} \quad f = \sum_{i=1}^{d_V} f(v_i) v^i.$$

Remark 2.2.3. Once chosen a right dual for every object X in \mathcal{C} we have that the assignment $(-)^* : \mathcal{C}^{\text{op}} \rightarrow \mathcal{C}$ defines a functor and $\text{ev} : (-) \boxtimes (-)^* \rightarrow \mathbb{I}$ and $\text{db} : \mathbb{I} \rightarrow (-)^* \boxtimes (-)$ define dinatural transformations³, i.e. for every X, Y and for all $f : X \rightarrow Y$ in \mathcal{C} we have

$$(f^* \boxtimes Y) \text{db}_Y = (X^* \boxtimes f) \text{db}_X \quad \text{and} \quad \text{ev}_X(X \boxtimes f^*) = \text{ev}_Y(f \boxtimes Y^*).$$

Let us assume that we have a different choice $(-)^{\vee} : \mathcal{C}^{\text{op}} \rightarrow \mathcal{C}$ of right dual objects. As a matter of notation, we will write $\text{ev}^{(*)}$ and $\text{db}^{(*)}$ to mean the evaluation and dual basis maps associated with the dual $(-)^*$ and $\text{ev}^{(\vee)}$ and $\text{db}^{(\vee)}$ to mean those associated with $(-)^{\vee}$. We know (see e.g. [M1, §9.3]) that for every X in \mathcal{C} , its right dual is unique up to isomorphism whenever it exists, i.e. we have an isomorphism $\kappa_X : X^* \rightarrow X^{\vee}$ in \mathcal{C} given by the composition

$$(28) \quad \kappa_X := \wp_{X^{\vee}} \left(X^{\vee} \boxtimes \text{ev}_X^{(*)} \right) \alpha_{X^{\vee}, X, X^*} \left(\text{db}_X^{(\vee)} \boxtimes X^* \right) \ell_{X^*}^{-1}.$$

The proof of the subsequent lemma is straightforward and it is left to the reader.

Lemma 2.2.4. *The isomorphism $\kappa_X : X^* \rightarrow X^{\vee}$ is natural in X and the dinatural transformations $\text{ev}^{(*)}$, $\text{db}^{(*)}$, $\text{ev}^{(\vee)}$ and $\text{db}^{(\vee)}$ satisfy*

$$(29) \quad (\kappa \boxtimes \text{id}) \text{db}^{(*)} = \text{db}^{(\vee)} \quad \text{and} \quad \text{ev}^{(\vee)} (\text{id} \boxtimes \kappa) = \text{ev}^{(*)}.$$

²Another terminology commonly used is *coevaluation*.

³More precisely, these should be referred to as *wedges*, since they are dinatural transformations to a constant functor. However, we avoided this in order to spare the proliferation of terminology. For the definition of dinatural transformations and wedges we refer to [ML, §9.4].

In what follows, unless stated otherwise, assume that a choice $(-)^*$ of dual object has been performed. Let us consider the following maps

$$(30) \quad \text{ev}_{\mathcal{F}(X)} := \varphi_0^{-1} \mathcal{F}(\text{ev}_X) \varphi_{X, X^*},$$

$$(31) \quad \text{db}_{\mathcal{F}(X)} := \varphi_{X^*, X}^{-1} \mathcal{F}(\text{db}_X) \varphi_0.$$

These do not endow $\mathcal{F}(X^*)$ with a structure of right dual object of $\mathcal{F}(X)$ in the category \mathfrak{M} because the functor $\mathcal{F} : \mathcal{C} \rightarrow \mathfrak{M}$ does not satisfy the associativity condition (2). Nevertheless, we have the following result, whose proof follows easily from the definitions and the dinaturality of ev and db .

Lemma 2.2.5. *The assignments $\text{ev}_{\mathcal{F}(X)}$ and $\text{db}_{\mathcal{F}(X)}$ defined in (30) and (31) give rise to dinatural transformations $\text{ev}_{\mathcal{F}(-)} : \mathcal{F} \otimes \mathcal{F}^* \rightarrow \mathbb{k}$ and $\text{db}_{\mathcal{F}(-)} : \mathbb{k} \rightarrow \mathcal{F}^* \otimes \mathcal{F}$.*

Remark 2.2.6. Recall that if $(\mathcal{F}, \phi, \phi_0) : (\mathcal{C}, \otimes, \mathbb{I}) \rightarrow (\mathcal{D}, \otimes, \mathbb{J})$ is a monoidal functor between monoidal categories and if X in \mathcal{C} has a right dual $(X^*, \text{ev}_X, \text{db}_X)$, then $\mathcal{F}(X)$ is right rigid with dual object $\mathcal{F}(X^*)$ and structure maps

$$(32) \quad \text{ev}_{\mathcal{F}(X)} = \phi_0^{-1} \mathcal{F}(\text{ev}_X) \phi_{X, X^*} \quad \text{and} \quad \text{db}_{\mathcal{F}(X)} = \phi_{X^*, X}^{-1} \mathcal{F}(\text{db}_X) \phi_0$$

(cf. e.g. [St, page 86]). Therefore, even if $\mathcal{F}(X^*)$ is not a right dual of $\mathcal{F}(X)$ in \mathfrak{M} , $(\mathcal{F}(X^*), \rho_{X^*})$ is a right dual of $(\mathcal{F}(X), \rho_X)$ in ${}^H\mathfrak{M}$ because $\mathcal{F}^H : \mathcal{C} \rightarrow {}^H\mathfrak{M}$ is monoidal. Evaluation and coevaluation maps are the same given in (30) and (31) and they are morphisms of comodules.

Let us pick an object X in \mathcal{C} . As a matter of notation, we are going to write

$$\text{db}_{\mathcal{F}(X)}(1_{\mathbb{k}}) = \sum_t \lambda^t \otimes x^t \in \mathcal{F}(X^*) \otimes \mathcal{F}(X)$$

and also $\gamma(x) := \text{ev}_{\mathcal{F}(X)}(x \otimes \gamma)$ for all $x \in \mathcal{F}(X)$ and $\gamma \in \mathcal{F}(X^*)$. Since $\mathcal{F}(X^*)$ is dual of $\mathcal{F}(X)$ in ${}^H\mathfrak{M}$, we may explicitly write (24) and (25) as follows:

$$(33a) \quad y = \sum_t \omega(y_{-1} \otimes \lambda_{-1}^t \otimes x_{-1}^t) \lambda_0^t(y_0) x_0^t,$$

$$(33b) \quad \gamma = \sum_t \omega^{-1}(\lambda_{-1}^t \otimes x_{-1}^t \otimes \gamma_{-1}) \gamma_0(x_0^t) \lambda_0^t,$$

for all $y \in \mathcal{F}(X)$ and for all $\gamma \in \mathcal{F}(X^*)$. Indeed, it's enough to track elements $y \in \mathcal{F}(X)$ and $\gamma \in \mathcal{F}(X^*)$ through the compositions in (24) and (25) respectively, keeping in mind relation (8).

Lemma 2.2.7. *We have natural transformations $\nu : \mathcal{F}(-)^* \rightarrow \mathcal{F}(-)^*$ and $\nu' : \mathcal{F}(-)^* \rightarrow \mathcal{F}(-)^*$ such that $\nu_X(\gamma)(x) = \gamma(x)$ and $\nu'_X(f) = \sum_t \lambda^t f(x^t)$ for all $x \in \mathcal{F}(X)$, $\gamma \in \mathcal{F}(X^*)$ and $f \in \mathcal{F}(X)^*$, where $(-)^*$ denotes the linear dual.*

Proof. The morphism $\nu_X : \mathcal{F}(X^*) \rightarrow \mathcal{F}(X)^*$ is defined as the composition

$$\mathcal{F}(X^*) \xrightarrow{\text{db}_{\mathcal{F}(X)}^{\mathbb{k}} \otimes \mathcal{F}(X^*)} \mathcal{F}(X)^* \otimes \mathcal{F}(X) \otimes \mathcal{F}(X^*) \xrightarrow{\mathcal{F}(X)^* \otimes \text{ev}_{\mathcal{F}(X)}} \mathcal{F}(X)^*,$$

where $\text{db}_{\mathcal{F}(X)}^{\mathbb{k}} : \mathbb{k} \rightarrow \mathcal{F}(X)^* \otimes \mathcal{F}(X)$ is the (twisted version of the) ordinary dual basis map for finite-dimensional vector spaces. The morphism the other way around, $\nu'_X : \mathcal{F}(X)^* \rightarrow \mathcal{F}(X^*)$, is given analogously as the composition

$$\mathcal{F}(X)^* \xrightarrow{\text{db}_{\mathcal{F}(X)} \otimes \mathcal{F}(X)^*} \mathcal{F}(X^*) \otimes \mathcal{F}(X) \otimes \mathcal{F}(X)^* \xrightarrow{\mathcal{F}(X^*) \otimes \text{ev}_{\mathcal{F}(X)}^{\mathbb{k}}} \mathcal{F}(X^*).$$

Naturality in X of both maps is a straightforward computation. \square

Remark 2.2.8. Notice that these are not inverses each other in general. Moreover, since $\mathcal{F}(X)^*$ does not have a natural structure of left H -comodule (it is a *right* H -comodule in fact), the linear maps ν_X and ν'_X cannot be seen as maps in ${}^H\mathfrak{M}$.

Lemma 2.2.9. *Let X be an object in \mathcal{C} and V in \mathfrak{M} . We have linear morphisms*

$$\phi_{X,V} : \mathcal{F}(X^*) \otimes V \rightarrow \text{Hom}(\mathcal{F}(X), V), \quad \psi_{X,V} : \text{Hom}(\mathcal{F}(X^*), V) \rightarrow V \otimes \mathcal{F}(X)$$

which are natural both in X and in V . Explicitly, for every generator $\gamma \otimes v \in \mathcal{F}(X^) \otimes V$, every $x \in \mathcal{F}(X)$ and every $f : \mathcal{F}(X^*) \rightarrow V$ we have*

$$\phi_{X,V}(\gamma \otimes v)(x) = \gamma(x)v \quad \text{and} \quad \psi_{X,V}(f) := \sum_t f(\lambda^t) \otimes x^t.$$

Proof. Recall that since $\mathcal{F}(X)$ is finite-dimensional, we have isomorphisms

$$\mathcal{F}(X)^* \otimes V \cong \text{Hom}(\mathcal{F}(X), V) \quad \text{and} \quad \text{Hom}(\mathcal{F}(X)^*, V) \cong V \otimes \mathcal{F}(X)$$

natural in V and X (in fact, they are the same up to a twist and the isomorphism $\mathcal{F}(X)^{**} \cong \mathcal{F}(X)$). If we pre-compose them with $\nu_X \otimes V$ and $\text{Hom}(\nu'_X, V)$ respectively, we find the natural transformations of the statement. \square

Lemma 2.2.10. *For every $V \in \mathfrak{M}$ we have natural bijections in Set*

$$\begin{aligned} \Phi_V : \text{Nat}(\mathcal{F}^*, \text{Hom}(\mathcal{F}, V)) &\rightarrow \text{Dinat}(\mathcal{F}^* \otimes \mathcal{F}, V), \\ \Psi_V : \text{Dinat}(\mathcal{F}^* \otimes \mathcal{F}, V) &\rightarrow \text{Nat}(\mathcal{F}, \text{Hom}(\mathcal{F}^*, V)). \end{aligned}$$

Proof. First of all, let us show that the statement makes sense. Recall that a coend $\int^X \mathcal{F}(X^*) \otimes \mathcal{F}(X)$ of the functor $\mathcal{F}^* \otimes \mathcal{F} : \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \mathfrak{M}$ is a dinatural transformation $\zeta : \mathcal{F}^* \otimes \mathcal{F} \rightarrow \int^X \mathcal{F}(X^*) \otimes \mathcal{F}(X)$, universal among dinatural transformations from $\mathcal{F}^* \otimes \mathcal{F}$ to a constant (see e.g. [ML, IX.6]). Since \mathfrak{M} is cocomplete and \mathcal{C} is essentially small, the coend $\int^X \mathcal{F}(X^*) \otimes \mathcal{F}(X)$ exists and we have a bijective correspondence

$$\text{Dinat}(\mathcal{F}^* \otimes \mathcal{F}, V) \cong \text{Hom}\left(\int^X \mathcal{F}(X^*) \otimes \mathcal{F}(X), V\right)$$

for every V . This implies that $\text{Dinat}(\mathcal{F}^* \otimes \mathcal{F}, V)$ is in fact a set. Once proven the existence of the bijective correspondences Φ_V and Ψ_V , we will have that both $\text{Nat}(\mathcal{F}^*, \text{Hom}(\mathcal{F}, V))$ and $\text{Nat}(\mathcal{F}, \text{Hom}(\mathcal{F}^*, V))$ are sets as well. In turn, the bijections are explicitly given by

$$\begin{aligned} \Phi_V(\nu)_X(\gamma \otimes x) &= \nu_X(\gamma)(x), & \Phi_V^{-1}(\delta)_X(\gamma)(x) &= \delta_X(\gamma \otimes x), \\ \Psi_V(\delta)_X(x)(\gamma) &= \delta_X(\gamma \otimes x), & \Psi_V^{-1}(\mu)_X(\gamma \otimes x) &= \mu_X(x)(\gamma), \end{aligned}$$

for every ν in $\text{Nat}(\mathcal{F}^*, \text{Hom}(\mathcal{F}, V))$, δ in $\text{Dinat}(\mathcal{F}^* \otimes \mathcal{F}, V)$, X in \mathcal{C} , $x \in \mathcal{F}(X)$ and $\gamma \in \mathcal{F}(X^*)$. Since checking that Φ_V^{-1} and Ψ_V^{-1} are in fact inverses of Φ_V and Ψ_V respectively and that Φ_V and Ψ_V are natural in V is analogous to the classical hom-tensor adjunction case, we will skip it. \square

2.2.11. The natural transformation ∇ . In view of the foregoing two lemmas, we may consider the chain of natural transformations

$$\begin{array}{ccc} \text{Nat}(\mathcal{F}^*, - \otimes \mathcal{F}^*) & & \text{Nat}(\mathcal{F}, \text{Hom}(\mathcal{F}^*, -)) \\ \searrow (\phi \circ \tau) \circ - & \nearrow \Psi \circ \Phi & \searrow \psi \circ - \\ & \text{Nat}(\mathcal{F}^*, \text{Hom}(\mathcal{F}, -)) & \text{Nat}(\mathcal{F}, - \otimes \mathcal{F}). \end{array}$$

The composition of these induces a natural transformation

$$\nabla^{\mathcal{F}} : \text{Nat}(\mathcal{F}, - \otimes \mathcal{F}) \rightarrow \text{Nat}(\mathcal{F}, - \otimes \mathcal{F})$$

that operates as follows. For every V in \mathfrak{M} and for every natural transformation $\xi \in \text{Nat}(\mathcal{F}, V \otimes \mathcal{F})$, write $\xi_X(y) = \sum y_V \otimes y_X$ for all X in \mathcal{C} and $y \in \mathcal{F}(X)$. We consider the natural transformation $\xi_\star := \xi(-)^\star$, where $(-)^\star : \mathcal{C}^{\text{op}} \rightarrow \mathcal{C}$ is the functor induced by the choice of the dual objects, and then we apply to it the above composition. Explicitly,

$$\nabla_V^{\mathcal{F}}(\xi)_X(y) = \sum_t \lambda_{X^\star}^t(y) \lambda_V^t \otimes x^t$$

for all X in \mathcal{C} , $y \in \mathcal{F}(X)$ and $\gamma \in \mathcal{F}(X^\star)$, i.e.

$$(34) \quad \nabla_V^{\mathcal{F}}(\xi)_X = (V \otimes \text{ev}_{\mathcal{F}(X)} \otimes \mathcal{F}(X)) \tau_{\mathcal{F}(X), V} \xi_{X^\star}(\mathcal{F}(X) \otimes \text{db}_{\mathcal{F}(X)}).$$

Lemma 2.2.12. *The natural transformation $\nabla^{\mathcal{F}}$ does not depend on the choice of the dual objects.*

Proof. Set $\nabla := \nabla^{\mathcal{F}}$ for shortness. Let us assume we have a different choice $(-)^\vee : \mathcal{C}^{\text{op}} \rightarrow \mathcal{C}$ of dual objects. As a matter of notation, we will write $\nabla^{(\star)}$ and $\nabla^{(\vee)}$ to mean the natural transformations induced by the functors $(-)^\star$ and $(-)^\vee$ respectively. Our aim is to show that $\nabla^{(\star)} = \nabla^{(\vee)}$ as natural transformations.

For every X in \mathcal{C} we have the natural isomorphism $\kappa_X : X^\star \rightarrow X^\vee$ given by (28). We will need the following relations, which descend easily from (29):

$$(35) \quad (\mathcal{F}(\kappa_X) \otimes \mathcal{F}(X)) \text{db}_{\mathcal{F}(X)}^{(\star)} = \text{db}_{\mathcal{F}(X)}^{(\vee)},$$

$$(36) \quad \text{ev}_{\mathcal{F}(X)}^{(\vee)}(\mathcal{F}(X) \otimes \mathcal{F}(\kappa_X)) = \text{ev}_{\mathcal{F}(X)}^{(\star)}.$$

Now, for every vector space V and for all $\xi \in \text{Nat}(\mathcal{F}, V \otimes \mathcal{F})$, set as above $\xi_\star := \xi(-)^\star$ and $\xi_\vee := \xi(-)^\vee$. Recall that we may write ξ_\star instead of $\mathcal{F}(X) \otimes \xi_\star \otimes \mathcal{F}(X)$. With this convention, for every object X in \mathcal{C} we may compute

$$\begin{aligned} \nabla_V^{(\vee)}(\xi)_X &\stackrel{(34)}{=} \left(V \otimes \text{ev}_{\mathcal{F}(X)}^{(\vee)} \otimes \mathcal{F}(X) \right) \tau_{\mathcal{F}(X), V} \xi_{X^\vee} \left(\mathcal{F}(X) \otimes \text{db}_{\mathcal{F}(X)}^{(\vee)} \right) \\ &\stackrel{(35)}{=} \left(V \otimes \text{ev}_{\mathcal{F}(X)}^{(\vee)} \otimes \mathcal{F}(X) \right) \tau_{\mathcal{F}(X), V} \xi_{X^\vee} \mathcal{F}(\kappa_X) \left(\mathcal{F}(X) \otimes \text{db}_{\mathcal{F}(X)}^{(\star)} \right) \\ &\stackrel{(*)}{=} \left(V \otimes \text{ev}_{\mathcal{F}(X)}^{(\vee)} \otimes \mathcal{F}(X) \right) \mathcal{F}(\kappa_X) \tau_{\mathcal{F}(X), V} \xi_{X^\star} \left(\mathcal{F}(X) \otimes \text{db}_{\mathcal{F}(X)}^{(\star)} \right) \\ &\stackrel{(36)}{=} \left(V \otimes \text{ev}_{\mathcal{F}(X)}^{(\star)} \otimes \mathcal{F}(X) \right) \tau_{\mathcal{F}(X), V} \xi_{X^\star} \left(\mathcal{F}(X) \otimes \text{db}_{\mathcal{F}(X)}^{(\star)} \right) \stackrel{(34)}{=} \nabla_V^{(\star)}(\xi)_X \end{aligned}$$

where in $(*)$ we used the naturality of ξ . This implies that $\nabla^{(\vee)} = \nabla^{(\star)}$, whence ∇ does not depend on the choice of the dual objects, as claimed. \square

2.3. Rigidity and the preantipode. In this section, finally, we show how it is possible to provide a preantipode for the coendomorphism coquasi-bialgebra of a right rigid monoidal category \mathcal{C} endowed with an "underlying" functor $\mathcal{F} : \mathcal{C} \rightarrow \mathfrak{M}_f$.

Remark 2.3.1. Since $\text{Hom}(H, -) \cong \text{Nat}(\mathcal{F}, - \otimes \mathcal{F})$, by Yoneda Lemma there exists a unique morphism $S \in \text{Hom}(H, H)$ such that the following commutes

$$(37) \quad \begin{array}{ccc} \text{Hom}(H, -) & \xrightarrow{\vartheta} & \text{Nat}(\mathcal{F}, - \otimes \mathcal{F}) \\ \text{Hom}(S, -) \downarrow & & \downarrow \nabla^{\mathcal{F}} \\ \text{Hom}(H, -) & \xrightarrow{\vartheta} & \text{Nat}(\mathcal{F}, - \otimes \mathcal{F}) \end{array}$$

From the commutativity of (37), if we take $V = H$ then we deduce that

$$(38) \quad (S \otimes \mathcal{F}) \rho^H = \nabla_H^{\mathcal{F}}(\rho^H).$$

For X in \mathcal{C} and $y \in \mathcal{F}(X)$, relation (38) rewrites explicitly as

$$(39) \quad \sum S(y_{-1}) \otimes y_0 = \sum_t \lambda_0^t(y) \lambda_{-1}^t \otimes x^t.$$

Furthermore, S can be characterized via relation (38), as Lemma 2.3.2 shows.

Lemma 2.3.2. *For $\sigma \in \text{Hom}(H, H)$, $\sigma = S$ if and only if $(\sigma \otimes \mathcal{F})\rho^H = \nabla_H^{\mathcal{F}}(\rho^H)$.*

Proof. Of course, the direct implication follows by definition of S . For the reverse implication observe that if V is any object in \mathfrak{M} and if $f \in \text{Hom}(H, V)$ then

$$\nabla_V^{\mathcal{F}}(\vartheta_V(f)) = \nabla_V^{\mathcal{F}}((f \otimes \mathcal{F}) \rho^H) = (f \otimes \mathcal{F}) \nabla_H^{\mathcal{F}}(\rho^H)$$

by naturality of ϑ and $\nabla^{\mathcal{F}}$. Hence if $(\sigma \otimes \mathcal{F}) \rho^H = \nabla_H^{\mathcal{F}}(\rho^H)$, then

$$\nabla_V^{\mathcal{F}}(\vartheta_V(f)) = (f \otimes \mathcal{F}) (\sigma \otimes \mathcal{F}) \rho^H = \vartheta_V(f \sigma) = (\vartheta_V \circ \text{Hom}(\sigma, V))(f)$$

and (37) commutes with $\text{Hom}(\sigma, -)$ instead of $\text{Hom}(S, -)$. \square

In different but equivalent words, S is the unique linear endomorphism of H such that $\vartheta_H(S) = \nabla_H^{\mathcal{F}}(\rho^H)$. Two important consequences may be drawn from this characterization: the uniqueness of the preantipode and the fact that any morphism of coquasi-bialgebras between coquasi-bialgebras with preantipodes automatically preserves them, as it happens in the Hopf algebra case.

Remark 2.3.3. Assume that B is a coquasi-bialgebra with a preantipode S_B . Denote by $\mathcal{U} : {}^B\mathfrak{M}_f \rightarrow \mathfrak{M}_f$ the forgetful functor and by $\rho^B \in \text{Nat}(\mathcal{U}, B \otimes \mathcal{U})$ the natural coaction of the B -comodules in ${}^B\mathfrak{M}_f$. It is shown in Appendix B that if V is an object in ${}^B\mathfrak{M}_f$, then an explicit right dual of V is given by $V^* = (V^* \otimes B)^{\text{co}B}$ with coaction $\rho_{V^*}^B(\sum_t f^t \otimes b^t) = \sum_t b_1^t \otimes (f^t \otimes b_2^t)$ and

$$\text{ev}_V \left(u \otimes \sum_t (f^t \otimes b^t) \right) = \sum_t f^t(u) \varepsilon(b^t), \quad \text{db}_V(1_{\mathbb{k}}) = \sum_{i=1}^{d_V} (v_0^i \otimes S_B(v_1^i)) \otimes v_i,$$

for all $\sum_t f^t \otimes b^t \in V^*$, $u \in V$ and where $\sum_{i=1}^{d_V} v^i \otimes v_i \in V^* \otimes V$ is a dual basis for V as a finite-dimensional vector space. In particular, ${}^B\mathfrak{M}_f$ is right rigid.

Corollary 2.3.4. *If a preantipode for a coquasi-bialgebra B exists, it is unique.*

Proof. Since ${}^B\mathfrak{M}_f$ is right rigid and $\mathcal{U} : {}^B\mathfrak{M}_f \rightarrow \mathfrak{M}_f$ is a tensor functor, we have a natural transformation $\nabla^{\mathcal{U}} : \text{Nat}(\mathcal{U}, - \otimes \mathcal{U}) \rightarrow \text{Nat}(\mathcal{U}, - \otimes \mathcal{U})$ as in 2.2.11. In view of Remark 2.3.3, we may compute explicitly for all V in ${}^B\mathfrak{M}_f$ and $y \in \mathcal{U}(V)$

$$\nabla_B^{\mathcal{U}}(\rho^B)_V(y) \stackrel{(34)}{=} ((B \otimes \text{ev}_V \otimes V) \tau_{V,B} \rho_{V^*}^B) \left(\sum_{i=1}^{d_V} y \otimes (v_0^i \otimes S_B(v_1^i)) \otimes v_i \right)$$

$$\begin{aligned}
&= \sum_{i=1}^{d_V} S_B(v_1^i)_1 v_0^i(y) \varepsilon(S_B(v_1^i)_2) \otimes v_i = \sum_{i=1}^{d_V} S_B(v_0^i(y) v_1^i) \otimes v_i \\
&= \sum_{i=1}^{d_V} S_B(y_{-1} v^i(y_0)) \otimes v_i = \sum S_B(y_{-1}) \otimes y_0,
\end{aligned}$$

so that $\nabla_B^\mathcal{U}(\rho^B) = (S_B \otimes \mathcal{U})\rho^B$. As B is a representing object for $\text{Nat}(\mathcal{U}, - \otimes \mathcal{U})$, this means that S_B satisfies condition (38) and so it follows from Lemma 2.3.2 that $S_B = S$, the unique linear endomorphism induced on B by $\nabla^\mathcal{F}$. \square

In order to prove the second result, we will need the following general result on the natural transformation ∇ .

Proposition 2.3.5. *Let $(\mathcal{C}, \boxtimes_{\mathcal{C}}, \mathbb{I}_{\mathcal{C}})$ and $(\mathcal{D}, \boxtimes_{\mathcal{D}}, \mathbb{I}_{\mathcal{D}})$ be essentially small right rigid monoidal categories. Let $(\mathcal{U}, \psi, \psi_0)$, $\mathcal{U} : \mathcal{D} \rightarrow \mathfrak{M}_f$, be a tensor functor and $(\mathcal{G}, \zeta, \zeta_0)$, $\mathcal{G} : \mathcal{C} \rightarrow \mathcal{D}$, be a monoidal one. For all $V \in \mathfrak{M}$ and $\eta \in \text{Nat}(\mathcal{U}, V \otimes \mathcal{U})$,*

$$(40) \quad \nabla_V^\mathcal{U}(\eta)\mathcal{G} = \nabla_V^{\mathcal{U}\mathcal{G}}(\eta\mathcal{G}).$$

Proof. Assume that we are given a choice of right duals $(-)^*$ in \mathcal{C} and $(-)^{\vee}$ in \mathcal{D} . Since \mathcal{G} is monoidal we have a natural isomorphism $\kappa_X : \mathcal{G}(X^*) \rightarrow \mathcal{G}(X)^{\vee}$ as in (28). Note that the composition $\mathcal{U}\mathcal{G} =: \mathcal{F}$ is still a tensor functor with structure isomorphisms $\phi = (\mathcal{U}\zeta) \circ \psi(\mathcal{G} \times \mathcal{G})$ and $\phi_0 = \mathcal{U}(\zeta_0)\psi_0$. Similarly to relations (35) and (36), one may check that

$$(41) \quad (\mathcal{U}\kappa \otimes \mathcal{U}\mathcal{G}) \circ \text{db}(\mathcal{U}\mathcal{G}) = (\text{db}\mathcal{U})\mathcal{G},$$

$$(42) \quad \text{ev}(\mathcal{U}\mathcal{G}) = (\text{ev}\mathcal{U})\mathcal{G} \circ (\mathcal{U}\mathcal{G} \otimes \mathcal{U}\kappa),$$

i.e., that for every object X in \mathcal{C} we have

$$(\mathcal{U}(\kappa_X) \otimes \mathcal{F}(X)) \text{db}_{\mathcal{F}(X)} = \text{db}_{\mathcal{U}(\mathcal{G}(X))}, \quad \text{ev}_{\mathcal{F}(X)}(\mathcal{F}(X) \otimes \mathcal{U}(\kappa_X^{-1})) = \text{ev}_{\mathcal{U}(\mathcal{G}(X))}.$$

As a consequence, for every $\eta \in \text{Nat}(\mathcal{U}, V \otimes \mathcal{U})$ we can compute

$$\begin{aligned}
\nabla_V^\mathcal{U}(\eta)_{\mathcal{G}(X)} &\stackrel{(34)}{=} (V \otimes \text{ev}_{\mathcal{U}(\mathcal{G}(X))} \otimes \mathcal{U}\mathcal{G}(X)) \tau_{\mathcal{U}\mathcal{G}(X), V} \eta_{\mathcal{G}(X)^{\vee}} (\mathcal{U}\mathcal{G}(X) \otimes \text{db}_{\mathcal{U}(\mathcal{G}(X))}) \\
&\stackrel{(42)}{=} (V \otimes \text{ev}_{\mathcal{U}\mathcal{G}(X)} \otimes \mathcal{U}\mathcal{G}(X)) \tau_{\mathcal{U}\mathcal{G}(X), V} \mathcal{U}(\kappa_X^{-1}) \eta_{\mathcal{G}(X)^{\vee}} (\mathcal{U}\mathcal{G}(X) \otimes \text{db}_{\mathcal{U}(\mathcal{G}(X))}) \\
&\stackrel{(*)}{=} (V \otimes \text{ev}_{\mathcal{U}\mathcal{G}(X)} \otimes \mathcal{U}\mathcal{G}(X)) \tau_{\mathcal{U}\mathcal{G}(X), V} \eta_{\mathcal{G}(X^*)} \mathcal{U}(\kappa_X^{-1}) (\mathcal{U}\mathcal{G}(X) \otimes \text{db}_{\mathcal{U}(\mathcal{G}(X))}) \\
&\stackrel{(41)}{=} (V \otimes \text{ev}_{\mathcal{U}\mathcal{G}(X)} \otimes \mathcal{U}\mathcal{G}(X)) \tau_{\mathcal{U}\mathcal{G}(X), V} \eta_{\mathcal{G}(X^*)} (\mathcal{U}\mathcal{G}(X) \otimes \text{db}_{\mathcal{U}\mathcal{G}(X)}) \stackrel{(34)}{=} \nabla_V^{\mathcal{U}\mathcal{G}}(\eta\mathcal{G})_X
\end{aligned}$$

where in $(*)$ we used the naturality of η . \square

Corollary 2.3.6. *Let $g : A \rightarrow B$ be a morphism between coquasi-bialgebras A and B with preantipodes S_A and S_B respectively. Then $gS_A = S_B g$.*

Proof. Since g is of coquasi-bialgebras, it induces a strict monoidal functor ${}^g\mathfrak{M} : {}^A\mathfrak{M} \rightarrow {}^B\mathfrak{M}$, which in turn restricts to a strict monoidal functor $\mathcal{G} : {}^A\mathfrak{M}_f \rightarrow {}^B\mathfrak{M}_f$ such that $\mathcal{V}\mathcal{G} = \mathcal{U}$, where $\mathcal{U} : {}^A\mathfrak{M}_f \rightarrow \mathfrak{M}_f$ and $\mathcal{V} : {}^B\mathfrak{M}_f \rightarrow \mathfrak{M}_f$ are the forgetful functors. Observe that, in particular, this implies that $(g \otimes \mathcal{U}(X))\rho_X^A = \rho_{\mathcal{G}(X)}^B$ for every X in ${}^A\mathfrak{M}_f$. Furthermore, the tensor structure on \mathcal{U} coincides with the one on the composition $\mathcal{V}\mathcal{G}$, hence we are fit to apply Proposition 2.3.5. Let us denote by $\xi : \text{Hom}(A, -) \rightarrow \text{Nat}(\mathcal{U}, - \otimes \mathcal{U})$ the natural isomorphism such that

$\xi_V(f) = (f \otimes \mathcal{U}) \rho^A$ for all V in \mathfrak{M} and $f \in \text{Hom}(A, V)$. We want to show that $\xi_B(g S_A) = \xi_B(S_B g)$. Omitting the constraints a, r, l , let us compute

$$\begin{aligned} \xi_B(g S_A) &= (g \otimes \mathcal{U})(S_A \otimes \mathcal{U}) \rho^A \stackrel{(38)}{=} (g \otimes \mathcal{U}) \nabla_A^{\mathcal{U}}(\rho^A) \stackrel{(*)}{=} \nabla_B^{\mathcal{U}}((g \otimes \mathcal{U}) \rho^A) \\ &= \nabla_B^{\mathcal{U}}(\rho^B \mathcal{G}) \stackrel{(40)}{=} \nabla_B^{\mathcal{V}}(\rho^B) \mathcal{G} = ((S_B \otimes \mathcal{V}) \rho^B) \mathcal{G} = (S_B \otimes \mathcal{V} \mathcal{G}) \rho^B \mathcal{G} \\ &= (S_B \otimes \mathcal{U})(g \otimes \mathcal{U}) \rho^A = \xi_B(S_B g) \end{aligned}$$

where in $(*)$ we used the naturality of $\nabla^{\mathcal{U}}$. Hence $g S_A = S_B g$ as desired. \square

Let us open a short digression, because it is worthy to notice that it is possible to give a direct proof as well of the result in Corollary 2.3.6. This will follow from Proposition 2.3.7 which was suggested by Alessandro Ardizzoni.

Proposition 2.3.7. *Let $(C, \Delta_C, \varepsilon_C)$ be a coalgebra, $(H, \Delta_H, \varepsilon_H, m, u, \omega, S)$ be a coquasi-bialgebra with preantipode S and let $g, h : C \rightarrow H$ be \mathbb{k} -linear maps such that g is a coalgebra morphism and g and h satisfies:*

$$(43) \quad \sum h(z_2)_1 \otimes g(z_1) h(z_2)_2 = h(z) \otimes 1,$$

$$(44) \quad \sum h(z_1)_1 g(z_2) \otimes h(z_1)_2 = 1 \otimes h(z),$$

$$(45) \quad \sum \omega(g(z_1) \otimes h(z_2) \otimes g(z_3)) = \varepsilon(z),$$

for all $z \in C$. Then $h = S g$.

Proof. As in [AP, §3.5], consider H itself as an element in ${}^H \mathfrak{M}^H$ but with trivial left coaction (we denote it by ${}^\circ H^\bullet$, where the empty dot denotes the trivial coaction and the full dot the coaction given by comultiplication) and the coquasi-Hopf bicomodule $H \hat{\otimes} H := {}^\circ H^\bullet \otimes {}^\bullet H_\bullet$ with explicit structures given by

$$\begin{aligned} \rho^r(x \otimes y) &= \sum x_1 \otimes y_1 \otimes x_2 y_2, & \rho^l(x \otimes y) &= \sum y_1 \otimes x \otimes y_2, \\ (x \otimes y) h &= \sum x_1 \otimes y_1 h_1 \omega(x_2 \otimes y_2 \otimes h_2) \end{aligned}$$

for all $x, y, h \in H$. Consider also the distinguished component $\hat{\varepsilon} : (H \hat{\otimes} H)^{\text{co}H} \otimes H \rightarrow H \hat{\otimes} H$ of the counit of the adjunction in §1.2, which is given explicitly by $\hat{\varepsilon}(x \otimes y \otimes h) = \sum x_1 \otimes y_1 h_1 \omega(x_2 \otimes y_2 \otimes h_2)$. Since H admits a preantipode, it is invertible with inverse $\hat{\varepsilon}^{-1}(x \otimes y) = \sum ((x_1 \otimes S(x_2)) \otimes x_3) y$ for all $x, y, h \in H$. Finally, consider the assignment $\beta : C \rightarrow H \otimes H \otimes H$ given by

$$\beta(z) = \sum g(z_1) \otimes h(z_2) \otimes g(z_3)$$

for all $z \in C$. Observe that

$$\begin{aligned} \rho^r\left(\sum g(z_1) \otimes h(z_2)\right) &= \sum g(z_1)_1 \otimes h(z_2)_1 \otimes g(z_1)_2 h(z_2)_2 \\ &\stackrel{(*)}{=} \sum g(z_1) \otimes h(z_3)_1 \otimes g(z_2) h(z_3)_2 \stackrel{(43)}{=} \sum g(z_1) \otimes h(z_2) \otimes 1, \end{aligned}$$

where in $(*)$ we used the hypothesis that g is comultiplicative, whence $\sum g(z_1) \otimes h(z_2) \in (H \hat{\otimes} H)^{\text{co}H}$ for all $z \in C$. Therefore for all $z \in C$ we can compute

$$\begin{aligned} \hat{\varepsilon}\beta(z) &= \hat{\varepsilon}\left(\sum g(z_1) \otimes h(z_2) \otimes g(z_3)\right) \\ &= \sum g(z_1)_1 \otimes h(z_2)_1 g(z_3)_1 \omega(g(z_1)_2 \otimes h(z_2)_2 \otimes g(z_3)_2) \end{aligned}$$

$$\begin{aligned}
&= \sum g(z_1) \otimes h(z_3)_1 g(z_4) \omega(g(z_2) \otimes h(z_3)_2 \otimes g(z_5)) \\
&\stackrel{(44)}{=} \sum g(z_1) \otimes 1_H \omega(g(z_2) \otimes h(z_3) \otimes g(z_4)) \stackrel{(45)}{=} g(z) \otimes 1_H
\end{aligned}$$

so that

$$\beta(z) = \hat{\varepsilon}^{-1}(g(z) \otimes 1_H) = \sum g(z)_1 \otimes S(g(z)_2) \otimes g(z)_3 = \sum g(z_1) \otimes S(g(z_2)) \otimes g(z_3)$$

and, by applying $\varepsilon \otimes H \otimes \varepsilon$ to both sides,

$$\begin{aligned}
h(z) &= \sum \varepsilon(g(z_1)) h(z_2) \varepsilon(g(z_3)) = (\varepsilon \otimes H \otimes \varepsilon) \left(\sum g(z_1) \otimes h(z_2) \otimes g(z_3) \right) \\
&= (\varepsilon \otimes H \otimes \varepsilon) (\beta(z)) = (\varepsilon \otimes H \otimes \varepsilon) \left(\sum g(z_1) \otimes S(g(z_2)) \otimes g(z_3) \right) = S(g(z)). \quad \square
\end{aligned}$$

Proof of Corollary 2.3.6. Since g is a morphism of coquasi-bialgebras, it is in particular a coalgebra morphism. Let $g S_A = h$ in Proposition 2.3.7 and compute

$$\begin{aligned}
\sum (g S_A(z_2))_1 \otimes g(z_1) (g S_A(z_2))_2 &= \sum g(S_A(z_2)_1) \otimes g(z_1 S_A(z_2)_2) = g S_A(z) \otimes 1, \\
\sum (g S_A(z_1))_1 g(z_2) \otimes (g S_A(z_1))_2 &= \sum g(S_A(z_1)_1 z_2) \otimes g(S_A(z_1)_2) = 1 \otimes g S_A(z), \\
\sum \omega_B(g(z_1) \otimes g S_A(z_2) \otimes g(z_3)) &= \sum \omega_A(z_1 \otimes S_A(z_2) \otimes z_3) = \varepsilon_A(z),
\end{aligned}$$

so that (43), (44) and (45) are satisfied. \square

The reason why we decided to retrieve this proof as well, apart from its shortness, is because it follows from the general result in Proposition 2.3.7, which is important in its own since it may find applications in other contexts. Going back to the main track, let us show finally that the morphism S induced by $\nabla^{\mathcal{F}}$ on the coendomorphism coquasi-bialgebra H is a preantipode.

Lemma 2.3.8. *For every $h \in H$ we have $\sum \omega(h_1 \otimes S(h_2) \otimes h_3) = \varepsilon(h)$.*

Proof. Recall from (33) that for every object X in \mathcal{C} and every $y \in \mathcal{F}(X)$

$$(46) \quad y = \sum_t \omega(y_{-1} \otimes \lambda_0^t(y_0) \lambda_{-1}^t \otimes x_{-1}^t) x_0^t,$$

as $\lambda_0^t(y_0) \in \mathbb{k}$. Omitting the associativity and unit constraints of the category of vector spaces, the right-hand member in (46) can be rewritten as

$$\begin{aligned}
&((\omega \otimes \mathcal{F}(X))(H \otimes H \otimes \rho_X)) \left(\sum y_{-1} \otimes \left(\sum_t \lambda_0^t(y_0) \lambda_{-1}^t \otimes x^t \right) \right) \\
&\stackrel{(39)}{=} ((\omega \otimes \mathcal{F}(X))(H \otimes H \otimes \rho_X)) \left(\sum y_{-1} \otimes \left(\sum S((y_0)_{-1}) \otimes (y_0)_0 \right) \right) \\
&= (\omega \otimes \mathcal{F}(X)) \left(\sum y_{-3} \otimes S(y_{-2}) \otimes y_{-1} \otimes y_0 \right) \\
&= \sum \omega(y_{-3} \otimes S(y_{-2}) \otimes y_{-1}) y_0.
\end{aligned}$$

For the sake of clearness, write $g(h) = \sum \omega(h_1 \otimes S(h_2) \otimes h_3)$ for all $h \in H$ so that $g \in \text{Hom}(H, \mathbb{k})$. Relation (46) rewrites now as

$$\vartheta_{\mathbb{k}}(\varepsilon)(y) = \sum \varepsilon(y_{-1}) y_0 = y = \sum g(y_{-1}) y_0 = \vartheta_{\mathbb{k}}(g)(y)$$

Since y was generic and $\vartheta_{\mathbb{k}}$ an isomorphism, this implies that $\varepsilon = g$ as claimed. \square

Lemma 2.3.9. *For every $h \in H$ we have $\sum S(h_1)_1 h_2 \otimes S(h_1)_2 = 1_H \otimes S(h)$.*

Proof. Since $\text{db}_{\mathcal{F}(X)}$ is a morphism of left H -comodules, we have explicitly that

$$(47) \quad \sum_t \lambda_{-1}^t x_{-1}^t \otimes \lambda_0^t \otimes x_0^t = 1_H \otimes \sum_t \lambda^t \otimes x^t$$

for every X in \mathcal{C} . Define $g : H \rightarrow H \otimes H$ by setting $g(h) = \sum S(h_1)_1 h_2 \otimes S(h_1)_2$ and analogously $g' : H \rightarrow H \otimes H$ by $g'(h) = 1_H \otimes S(h)$. We want to show that $\vartheta_{H \otimes H}(g) = \vartheta_{H \otimes H}(g')$. For all X in \mathcal{C} and $y \in \mathcal{F}(X)$, we have on the one hand

$$\begin{aligned} \vartheta_{H \otimes H}(g)_X(y) &= (g \otimes \mathcal{F}(X)) \left(\sum y_{-1} \otimes y_0 \right) = \sum S(y_{-2})_1 y_{-1} \otimes S(y_{-2})_2 \otimes y_0 \\ &= \text{coact}_{H \otimes \mathcal{F}(X)} \left(\sum S(y_{-1}) \otimes y_0 \right) = \text{coact}_{H \otimes \mathcal{F}(X)} (\vartheta_H(S)_X(y)), \end{aligned}$$

and on the other hand

$$\vartheta_{H \otimes H}(g')_X(y) = 1_H \otimes \sum S(y_{-1}) \otimes y_0 = 1_H \otimes \vartheta_H(S)_X(y).$$

Therefore, $\vartheta_{H \otimes H}(g)_X = \vartheta_{H \otimes H}(g')_X$ if and only if $\vartheta_H(S)_X(y)$ lies in the left coinvariant elements of $H \otimes \mathcal{F}(X)$ for every $y \in \mathcal{F}(X)$. Writing explicitly this condition by mean of (39), we find out that it is equivalent to require that

$$(48) \quad \begin{aligned} 1_H \otimes \sum_t \lambda_0^t(y) \lambda_{-1}^t \otimes x^t &= \sum_t (\lambda_{-1}^t)_1 x_{-1}^t \otimes \lambda_0^t(y) (\lambda_{-1}^t)_2 \otimes x_0^t \\ &= \sum_t \lambda_{-2}^t x_{-1}^t \otimes \lambda_0^t(y) \lambda_{-1}^t \otimes x_0^t. \end{aligned}$$

However, by applying $H \otimes (\phi_{X,H} \tau_{H,\mathcal{F}(X^*)} \rho_{X^*}) \otimes \mathcal{F}(X)$ to both sides of (47), where ϕ is the same of Lemma 2.2.9, we get that for all $y \in \mathcal{F}(X)$

$$\sum_t \lambda_{-2}^t x_{-1}^t \otimes \lambda_0^t(-) \lambda_{-1}^t \otimes x_0^t = 1_H \otimes \sum_t \lambda_0^t(-) \lambda_{-1}^t \otimes x^t$$

in $H \otimes \text{Hom}(\mathcal{F}(X), H) \otimes \mathcal{F}(X)$ so that (48) holds and $\vartheta_{H \otimes H}(g)_X = \vartheta_{H \otimes H}(g')_X$ for every object X in \mathcal{C} . By bijectivity of $\vartheta_{H \otimes H}$, we get $g = g'$ in $\text{Hom}(H, H \otimes H)$, whence $\sum S(h_1)_1 h_2 \otimes S(h_1)_2 = 1_H \otimes S(h)$ for every $h \in H$. \square

Lemma 2.3.10. *For every $h \in H$ we have $\sum S(h_2)_1 \otimes h_1 S(h_2)_2 = S(h) \otimes 1_H$.*

Proof. As above, for every X in \mathcal{C} , $y \in \mathcal{F}(X)$ and $\gamma \in \mathcal{F}(X^*)$ we have

$$(49) \quad \sum y_{-1} \gamma_{-1} \gamma_0(y_0) = \gamma(y) 1_H,$$

since the morphism $\text{ev}_{\mathcal{F}(X)}$ is left H -colinear. Consider $g, g' : H \rightarrow H \otimes H$ given by $g(h) = \sum S(h_2)_1 \otimes h_1 S(h_2)_2$ and $g'(h) = S(h) \otimes 1_H$ respectively. We want to prove that $\vartheta_{H \otimes H}(g) = \vartheta_{H \otimes H}(g')$, i.e. that for every X in \mathcal{C} and $y \in \mathcal{F}(X)$

$$\sum S(y_{-1})_1 \otimes y_{-2} S(y_{-1})_2 \otimes y_0 = \sum S(y_{-1}) \otimes 1_H \otimes y_0.$$

To this aim, note that for every X in \mathcal{C} and $y \in \mathcal{F}(X)$ we have on the one hand

$$\begin{aligned} &\sum S(y_{-1})_1 \otimes y_{-2} S(y_{-1})_2 \otimes y_0 \\ &= (((H \otimes m) (\tau_{H,H} \otimes H) (H \otimes \Delta)) \otimes \mathcal{F}(X)) \left(\sum y_{-2} \otimes S(y_{-1}) \otimes y_0 \right) \\ &\stackrel{(39)}{=} (((H \otimes m) (\tau_{H,H} \otimes H) (H \otimes \Delta)) \otimes \mathcal{F}(X)) \left(\sum y_{-1} \otimes \lambda_0^t(y_0) \lambda_{-1}^t \otimes x^t \right) \\ &= \sum \lambda_{-2}^t \otimes y_{-1} \lambda_{-1}^t \lambda_0^t(y_0) \otimes x^t \end{aligned}$$

and on the other hand

$$\begin{aligned}
\sum S(y_{-1}) \otimes 1_H \otimes y_0 &= ((H \otimes u \otimes \mathcal{F}(X))(H \otimes l_{\mathcal{F}(X)}^{-1})) \left(\sum S(y_{-1}) \otimes y_0 \right) \\
&\stackrel{(39)}{=} ((H \otimes u \otimes \mathcal{F}(X))(H \otimes l_{\mathcal{F}(X)}^{-1})) \left(\sum_t \lambda_0^t(y) \lambda_{-1}^t \otimes x^t \right) \\
&= \sum_t \lambda_{-1}^t \otimes \lambda_0^t(y) 1_H \otimes x^t.
\end{aligned}$$

Relation (49) implies that the two latest terms of the foregoing computations are equal. Therefore $\vartheta_{H \otimes H}(g)_X = \vartheta_{H \otimes H}(g')_X$ for every object X in \mathcal{C} , from which it follows that $g = g' \in \text{Hom}(H, H \otimes H)$ by bijectivity of $\vartheta_{H \otimes H}$. Thus $\sum S(h_2)_1 \otimes h_1 S(h_2)_2 = S(h) \otimes 1_H$ for every $h \in H$. \square

Summing up, we have the following central result, connecting the rigidity of the category \mathcal{C} with the existence of a preantipode for the coendomorphism coquasi-bialgebra H .

Theorem 2.3.11. *In the hypotheses of Theorem 2.1.5, if furthermore \mathcal{C} is right rigid, then H admits a preantipode. Moreover, if B is another coquasi-bialgebra with preantipode such that \mathcal{F} factorizes through a monoidal functor $\mathcal{G} : \mathcal{C} \rightarrow {}^B\mathfrak{M}$ followed by the forgetful functor, then the unique coquasi-bialgebra morphism $\epsilon : H \rightarrow B$ provided by Theorem 2.1.5 preserves the preantipodes.*

Proof. The existence of a preantipode S for H follows from Remark 2.3.1 and Lemmas 2.3.8, 2.3.9, 2.3.10. If B is another coquasi-bialgebra with preantipode as in the statement, then Majid's Theorem 2.1.5 implies that there is a unique map of coquasi-bialgebras $\epsilon : H \rightarrow B$ inducing a functor $\epsilon\mathfrak{M} : {}^H\mathfrak{M} \rightarrow {}^B\mathfrak{M}$ such that $\epsilon\mathfrak{M}\mathcal{F}^H = \mathcal{G}$. In view of Corollary 2.3.6, ϵ preserves the preantipodes. \square

2.4. Applications. Let us conclude with some consequences of Theorem 2.3.11. The first one is the well-known result of Ulbrich [U] about Hopf algebras and rigid monoidal categories, in the case of a base field instead of a commutative ring.

Example 2.4.1 (see [U, page 255, Theorem]). *Let \mathcal{C} be an essentially small right rigid monoidal category and let $\mathcal{F} : \mathcal{C} \rightarrow \mathfrak{M}_f$ be a monoidal functor. Then the coendomorphism coquasi-bialgebra H provided by Theorem 2.1.5 is a bialgebra (i.e. $\omega = \varepsilon \otimes \varepsilon \otimes \varepsilon$) and the preantipode provided by Theorem 2.3.11 satisfies*

$$\varepsilon S(h) = \sum \omega(h_1 \otimes S(h_2) \otimes h_3) \stackrel{(13)}{=} \varepsilon(h),$$

i.e. it is an ordinary antipode (see Remark 1.2.2) and H is a Hopf algebra.

The second immediate consequence is the fact that any (coquasi-)Hopf algebra is in particular a coquasi-bialgebra with preantipode.

Definition 2.4.2 (cf. [M1, page 66] or [Sc1, page 333]). *A coquasi-Hopf algebra $(H, m, u, \Delta, \varepsilon, \omega, s, \alpha, \beta)$ is a coquasi-bialgebra H endowed with a coquasi-antipode (s, α, β) , i.e. a coalgebra anti-homomorphism $s : H \rightarrow H$ and two maps α, β in H^* , such that, for all $h \in H$*

$$\begin{aligned}
\sum h_1 \beta(h_2) s(h_3) &= \beta(h) 1_H, \quad \sum s(h_1) \alpha(h_2) h_3 = \alpha(h) 1_H, \\
\sum \omega(h_1 \otimes \beta(h_2) s(h_3) \alpha(h_4) \otimes h_5) &= \varepsilon(h),
\end{aligned}$$

$$\sum \omega^{-1}(s(h_1) \otimes \alpha(h_2)h_3\beta(h_4) \otimes s(h_5)) = \varepsilon(h).$$

Example 2.4.3 ([AP, Theorem 3.10]). *Let H be a coquasi-Hopf algebra with coquasi-antipode (s, α, β) . Analogously to the Hopf algebra case, the category of finite-dimensional left H -comodules ${}^H\mathfrak{M}_f$ is a right rigid monoidal category, where the dual of (V, ρ) in ${}^H\mathfrak{M}_f$ is given by its dual vector space V^* with comodule structure $\rho_*(f) = \sum f_{-1} \otimes f_0$ given by the relation*

$$(50) \quad \sum f_{-1}f_0(v) = \sum s(v_{-1})f(v_0)$$

for all $v \in V$, $f \in V^*$. In details, if $\text{db}_V^{\mathbb{k}}(1_{\mathbb{k}}) = \sum_{i=1}^{d_V} v^i \otimes v_i \in V^* \otimes V$ is a dual basis for V as a vector space, then

$$\rho_*(f) = \sum_{i=1}^{d_V} s((v_i)_{-1})f((v_i)_0) \otimes v^i$$

and the evaluation and dual basis morphisms are given by

$$\text{ev}_V(v \otimes f) = \sum \beta(v_{-1})f(v_0), \quad \text{db}_V(1_{\mathbb{k}}) = \sum_{i=1}^{d_V} v^i \otimes \alpha((v_i)_{-1})(v_i)_0$$

for all $v \in V$, $f \in V^*$ (cf. [Sc1, page 334]). If we consider the forgetful functor $\mathcal{U} : {}^H\mathfrak{M}_f \rightarrow \mathfrak{M}_f$, this is a tensor functor so that we may apply Theorem 2.3.11. The outcome is a coquasi-bialgebra with preantipode structure on H , where ω is the former one and S is uniquely determined by (39). Note that from $\sum_t \lambda^t \otimes x^t = \sum_{i=1}^{d_V} v^i \otimes \alpha((v_i)_{-1})(v_i)_0$ it follows that

$$\begin{aligned} \sum_t \lambda_0^t(y) \lambda_{-1}^t \otimes x^t &= \sum_{i=1}^{d_V} \beta(y_{-1})v_0^i(y_0)v_{-1}^i \otimes \alpha((v_i)_{-1})(v_i)_0 \\ &\stackrel{(50)}{=} \sum_{i=1}^{d_V} \beta(y_{-2})s(y_{-1})v^i(y_0) \otimes \alpha((v_i)_{-1})(v_i)_0 \\ &= \sum_{i=1}^{d_V} \beta(y_{-2})s(y_{-1}) \otimes \alpha((v_i)_{-1})v^i(y_0)(v_i)_0 \\ &= \sum \beta(y_{-3})s(y_{-2})\alpha(y_{-1}) \otimes y_0. \end{aligned}$$

Therefore, $S = \beta * s * \alpha$ where $*$ is the convolution product.

As a final application of the theory we developed, let us show that the finite dual coalgebra of a quasi-bialgebra with preantipode is a coquasi-bialgebra with preantipode (for the definition of the finite dual coalgebra we refer to [Sw, Chapter VI], for the definition of a preantipode for a quasi-bialgebra and its properties we refer to [Sa]). The proof of this fact lies on the following result, which we suppose should be well-known, but we were not able to find a reference.

Lemma 2.4.4. *Let A be an algebra and A° be its finite dual coalgebra. We have an isomorphism $\mathcal{L} : {}^{A^\circ}\mathfrak{M}_f \rightarrow {}_f\mathfrak{M}_A$ between the category of finite-dimensional left A° -comodules and that of finite-dimensional right A -modules that satisfies ${}^{A^\circ}\mathcal{U}\mathcal{L} = \mathcal{U}_A$, where \mathcal{U}_A and ${}^{A^\circ}\mathcal{U}$ are the obvious forgetful functors.*

Proof. Consider the algebra morphism $\pi : A \rightarrow A^{\circ*}$ induced by the injection $A^{\circ} \hookrightarrow A^*$ as in [Sw, page 117]. Since A° is a coalgebra, we have a functor $\mathcal{L} : {}^{A^{\circ}}\mathfrak{M} \rightarrow \mathfrak{M}_A$ given by the composition

$${}^{A^{\circ}}\mathfrak{M} \longrightarrow \mathfrak{M}_{A^{\circ*}} \xrightarrow{\mathfrak{M}_{\pi}} \mathfrak{M}_A.$$

In details, to every left A° -comodule (N, ρ_N) it associates the right A -module (N, μ_N^{ρ}) with action given by

$$\mu_N^{\rho}(n \otimes a) = \sum n_{-1}(a)n_0.$$

This functor satisfies $\mathcal{U}_A \mathcal{L} = {}^{A^{\circ}}\mathcal{U}$ and it restricts to a functor ${}^{A^{\circ}}\mathfrak{M}_f \rightarrow {}_f\mathfrak{M}_A$ between finite-dimensional A° -comodules and finite-dimensional A -modules, which we denote by \mathcal{L} again. The other way around, we have a functor $\mathcal{R} : {}_f\mathfrak{M}_A \rightarrow {}^{A^{\circ}}\mathfrak{M}_f$ which assigns to every finite-dimensional right A -module (M, μ_M) , the left A° -comodule (M, ρ_M^{μ}) with coaction given by

$$(51) \quad \rho_M^{\mu}(m) = \sum_{i=1}^{d_M} (e^i \mu_m) \otimes e_i$$

where $\mu_m : A \rightarrow M$ is the linear map that sends a to $\mu_m(m \otimes a)$ and $\sum_{i=1}^{d_M} e^i \otimes e_i \in M^* \otimes M$ is a dual basis for M as a vector space. It can be checked that $\text{im}(\rho_M^{\mu}) \subseteq A^{\circ} \otimes M$, whence (M, ρ_M^{μ}) is in fact a finite-dimensional left A° -comodule, and that the functors \mathcal{L} and \mathcal{R} satisfy $\mathcal{L}\mathcal{R} = \text{Id}$ and $\mathcal{R}\mathcal{L} = \text{Id}$, so that they give the desired isomorphism. Notice that ${}^{A^{\circ}}\mathcal{U}\mathcal{R} = \mathcal{U}_A$ as well. \square

Lemma 2.4.5. *Let $(A, m, u, \Delta, \varepsilon, \Phi, S)$ be a quasi-bialgebra with preantipode. The category of finite-dimensional right A -modules ${}_f\mathfrak{M}_A$ is a right rigid monoidal category with tensor forgetful functor $\mathcal{U}_A : {}_f\mathfrak{M}_A \rightarrow \mathfrak{M}_f$.*

Proof. As it happens for coquasi-bialgebras, the axioms of a quasi-bialgebra ensures that the category of right A -modules \mathfrak{M}_A becomes a monoidal category with tensor product the tensor product over \mathbb{k} , unit object \mathbb{k} itself, and associativity constraint

$$\alpha_{M,N,P}((m \otimes n) \otimes p) = (m \otimes (n \otimes p)) \cdot \Phi^{-1}.$$

for all M, N, P in \mathfrak{M}_A and m, n, p in M, N, P respectively. The unit constraints are the same of \mathfrak{M} . In particular, the forgetful functor $\mathcal{U}_A : \mathfrak{M}_A \rightarrow \mathfrak{M}$ is a tensor functor and the same property holds for its restriction to finite-dimensional modules. By the right-handed version of [Sa, Theorem 4], the existence of a preantipode for A is equivalent to a structure theorem for quasi-Hopf A -bimodules. As a consequence, it is possible to replicate the same procedure of Appendix B to show that ${}_f\mathfrak{M}_A$ is right rigid. Otherwise, one may check directly that a dual object of an A -module M is given by the right A -module

$$M^{\star} := \frac{A \otimes M^*}{A^+ (A \otimes M^*)}$$

where $A^+ := \ker(\varepsilon)$ and M^* is the \mathbb{k} -linear dual of M . The A -module structure on M^{\star} is $\overline{a \otimes f} \cdot x = \overline{ax \otimes f}$ for all $a, x \in A$ and $f \in M^*$. Evaluation and dual basis maps are explicitly given by

$$\text{ev}_M(m \otimes \overline{a \otimes f}) = f(m \cdot S(a)) \quad \text{and} \quad \text{db}_M(1_{\mathbb{k}}) = \sum_{i=1}^{d_M} \overline{1_A \otimes e^i} \otimes e_i$$

for all $m \in M$, $f \in M^*$ and $a \in A$ and where $\sum_{i=1}^{d_M} e^i \otimes e_i$ is a dual basis of M as a finite-dimensional vector space. \square

Proposition 2.4.6. *Let $(A, m, u, \Delta, \varepsilon, \Phi, S)$ be a quasi-bialgebra with preantipode. Let $(A^\circ, \Delta_\circ, \varepsilon_\circ)$ be its finite dual coalgebra. Then A° can be endowed with a structure of a coquasi-bialgebra with preantipode.*

Proof. Denote by $\mathcal{U} : {}_f\mathfrak{M}_A \rightarrow \mathfrak{M}_f$ and $\mathcal{V} : {}^{A^\circ}\mathfrak{M}_f \rightarrow \mathfrak{M}_f$ the forgetful functors. As a consequence of Lemma 2.4.4, we have a chain of natural isomorphism

$$\text{Nat}(\mathcal{U}, - \otimes \mathcal{U}) \cong \text{Nat}(\mathcal{V}, - \otimes \mathcal{V}) \cong \text{Hom}(A^\circ, -)$$

which allows us to consider A° itself as a representing object for $\text{Nat}(\mathcal{U}, - \otimes \mathcal{U})$. If we consider then the category of finite-dimensional right A -modules ${}_f\mathfrak{M}_A$ as a right rigid monoidal category together with the tensor forgetful functor $\mathcal{U} : {}_f\mathfrak{M}_A \rightarrow \mathfrak{M}_f$, then A° can be endowed with a structure of a coquasi-bialgebra with preantipode in view of Theorem 2.3.11. \square

Remark 2.4.7. It is worthy to point out that the corestriction $\mathcal{U}^{A^\circ} : {}_f\mathfrak{M}_A \rightarrow {}^{A^\circ}\mathfrak{M}_f$ of the functor $\mathcal{U}^{A^\circ} : {}_f\mathfrak{M}_A \rightarrow {}^{A^\circ}\mathfrak{M}$ provided by Theorem 2.3.11 coincides with the functor \mathcal{R} , which becomes this way a strict monoidal functor.

Remark 2.4.8. If we want to know explicitly the coquasi-bialgebra structure on A° we may proceed as follows. First of all observe that the tensor structure on $\mathcal{U} : {}_f\mathfrak{M}_A \rightarrow \mathfrak{M}_f$ is the strict one: $\varphi_{M,N} = \text{id}_{M \otimes N}$ and $\varphi_0 = \text{id}_{\mathbb{k}}$. Secondly, for every object M in ${}_f\mathfrak{M}_A$ the natural transformation $\rho_M : \mathcal{U}(M) \rightarrow A^\circ \otimes \mathcal{U}(M)$ is given by the coaction (51). Let us denote by $\sum_{i=1}^{d_M} e_M^i \otimes e_i^M \in M^* \otimes M$ a dual basis for M as a vector space, for all M in ${}_f\mathfrak{M}_A$. If we denote by $\mu^{M \otimes N}$ the A -action on the tensor product, then

$$\rho_{M \otimes N}(x) = \sum_{i,j} ((e_M^i \otimes e_N^j) \mu_x^{M \otimes N}) \otimes (e_i^M \otimes e_j^N)$$

for all $x \in M \otimes N$, where we considered $M^* \otimes N^*$ injected in $(M \otimes N)^*$. Furthermore, it is well-known from the associative case that the convolution product $*$, given by $(f * g)(a) = \sum f(a_1)g(a_2)$ for all $f, g \in A^*$ and $a \in A$, restricts to a morphism $*$: $A^\circ \otimes A^\circ \rightarrow A^\circ$. It is also clear that $\varepsilon \in A^\circ$. To show that they are the multiplication and the unit induced on A° , denote by μ^M and μ^N the A -actions on M and N respectively and compute for $\sum_{i=1}^t m_i \otimes n_i \in M \otimes N$

$$(A^\circ \otimes \varphi_{M,N}) \left(\vartheta_{A^\circ}^2(*)_{M,N} \left(\sum_{i=1}^t m_i \otimes n_i \right) \right) = \sum_{i,h,k} ((e_M^h \mu_{m_i}^M) * (e_N^k \mu_{n_i}^N)) \otimes (e_h^M \otimes e_k^N).$$

Since for every $a \in A$, $f \in M^*$, $g \in N^*$ and $x = \sum_{i=1}^t m_i \otimes n_i \in M \otimes N$ we have

$$\sum_{i=1}^t ((f \mu_{m_i}^M) * (g \mu_{n_i}^N))(a) = \sum_{i=1}^t (f \mu_{m_i}^M)(a_1) (g \mu_{n_i}^N)(a_2) = (f \otimes g) \mu_x^{M \otimes N}(a),$$

we conclude that $(A^\circ \otimes \varphi_{M,N}) \vartheta_{A^\circ}^2(*)_{M,N} = \rho_{M \otimes N} \varphi_{M,N}$ and by uniqueness of the morphism $A^\circ \otimes A^\circ \rightarrow A^\circ$ satisfying this relation we have that the multiplication induced on A° is exactly $*$. Moreover, if we compute

$$r_{A^\circ}(\rho_{\mathbb{k}}(1_{\mathbb{k}})) = r_{A^\circ}(\varepsilon \otimes 1_{\mathbb{k}}) = \varepsilon,$$

then we recover that the unit of the multiplication $*$ is ε , in view of (21) and the fact that $\varphi_0 = \text{id}_{\mathbb{k}}$. Consider also the assignment

$$\omega : A^\circ \otimes A^\circ \otimes A^\circ \rightarrow \mathbb{k}; \quad \omega(f \otimes g \otimes h) = \sum f(\Phi^1) g(\Phi^2) h(\Phi^3).$$

For every M, N, P in ${}_f\mathfrak{M}_A$ and all $m \in M, n \in N, p \in P$, it satisfies

$$\begin{aligned} & \varphi_{M \otimes N, P}((\varphi_{M, N} \otimes \mathcal{U}(P))(\vartheta_{\mathbb{k}}^3(\omega)_{M, N, P}(m \otimes n \otimes p))) \\ &= \sum_{i, j, k} \omega((e_M^i \mu_m^M) \otimes (e_N^j \mu_n^N) \otimes (e_P^k \mu_p^P)) e_i^M \otimes e_j^N \otimes e_k^P \\ &= \sum m \cdot \Phi^1 \otimes n \cdot \Phi^2 \otimes p \cdot \Phi^3, \end{aligned}$$

whence $\varphi_{M \otimes N, P}(\varphi_{M, N} \otimes \mathcal{U}(P))\vartheta_{\mathbb{k}}^3(\omega)_{M, N, P} = \mathcal{U}_A(\alpha_{M, N, P}^{-1})\varphi_{M, N \otimes P}(\mathcal{U}(M) \otimes \varphi_{N, P})$ and so ω is in fact the induced reassociator. The antipode can be constructed explicitly as well. Consider the transpose $S^* : A^* \rightarrow A^*$. Let us show firstly that S^* factors through a linear map $S^\circ : A^\circ \rightarrow A^\circ$; the proof relies on formula (59) from Appendix A. Pick $f \in A^\circ$ and compute

$$\begin{aligned} S^*(f)(ab) &= f(S(ab)) \stackrel{(59)}{=} \sum f(S(\varphi^1 b) \varphi^2 S(\psi^1 \varphi^3) \psi^2 S(a \psi^3)) \\ &= \sum f_1 S(\varphi^1 b) f_2 (\varphi^2 S(\psi^1 \varphi^3) \psi^2) f_3 S(a \psi^3) \\ &= \left(\sum (\psi^3 \rightharpoonup f_3 S) \otimes f_2 (\varphi^2 S(\psi^1 \varphi^3) \psi^2) (f_1 S \leftarrow \varphi^1) \right) (a \otimes b). \end{aligned}$$

Since this implies that $m^*(S^*(f)) \in A^* \otimes A^*$, in view of [Sw, Proposition 6.0.3] we have that $S^*(f) \in A^\circ$. Let us prove now that S° satisfies the relation $\vartheta_{A^\circ}(S^\circ) = \nabla_{A^\circ}^{\mathcal{U}_A}(\rho^{A^\circ})$. For all M in ${}_f\mathfrak{M}_A$ and all $m \in M$ we need to show that

$$(52) \quad \sum S^\circ(m_{-1}) \otimes m_0 = \sum_{i=1}^{d_M} \left(\overline{1_A \otimes e^i} \right)_0(m) \left(\overline{1_A \otimes e^i} \right)_{-1} \otimes e_i.$$

Since M^* is finite-dimensional, we may fix a dual basis $\sum_{j=1}^{d_{M^*}} \gamma_j \otimes \gamma_j$ of M^* as an object in \mathfrak{M}_f and then the right-hand member of (52) can be rewritten as

$$\sum_{i=1}^{d_M} \sum_{j=1}^{d_{M^*}} \gamma_j(m) \left(\gamma_j \mu_{1_A \otimes e^i}^{M^*} \right) \otimes e_i$$

in view of (51). Let us focus on $\sum_{j=1}^{d_{M^*}} \gamma_j(m) \left(\gamma_j \mu_{1_A \otimes e^i}^{M^*} \right) \in A^\circ$. For all $a \in A$,

$$\sum_{j=1}^{d_{M^*}} \gamma_j(m) \left(\gamma_j \mu_{1_A \otimes e^i}^{M^*} \right) (a) = \sum_{j=1}^{d_{M^*}} \gamma_j(m) \gamma_j \left(\overline{a \otimes e^i} \right) = \overline{a \otimes e^i}(a) = e^i(m \cdot S(a))$$

and since $e^i(m \cdot S(a)) = S^\circ(e^i \mu_m^M)(a)$, we have

$$\sum_{i=1}^{d_M} \sum_{j=1}^{d_{M^*}} \gamma_j(m) \left(\gamma_j \mu_{1_A \otimes e^i}^{M^*} \right) \otimes e_i = \sum_i S^\circ(e^i \mu_m^M) \otimes e_i.$$

We can conclude then that relation (52) is satisfied, as desired.

Remark 2.4.9. The fact that the finite dual coalgebra of a quasi-bialgebra can be endowed with a structure of coquasi-bialgebra has already been shown in [AES, §5.2] with a different approach. Moreover, the fact that the coendomorphism coalgebra of the forgetful functor $\mathcal{U} : {}_f\mathfrak{M}_A \rightarrow \mathfrak{M}_f$ is isomorphic to the finite dual coalgebra A° seems to be well-known (cf. e.g. [EG, Example 4.11]), however again we didn't find an explicit reference for this.

APPENDIX A. A RELATION FOR THE PREANTIPODE OF A QUASI-BIALGEBRA

Recall from [Sa] that a preantipode for a quasi-bialgebra $(A, \Delta, \varepsilon, m, u, \Phi)$ is a \mathbb{k} -linear map $S : A \rightarrow A$ that satisfies

$$(53) \quad \sum a_1 S(ba_2) = \varepsilon(a) S(b) = \sum S(a_1 b) a_2, \quad \sum \Phi^1 S(\Phi^2) \Phi^3 = 1,$$

for all $a, b \in A$, where $\sum \Phi^1 \otimes \Phi^2 \otimes \Phi^3 = \Phi$. Let us introduce also the following extended notation for the reassociator and its inverse:

$$\begin{aligned} \Phi &= \sum \Phi^1 \otimes \Phi^2 \otimes \Phi^3 = \sum \Psi^1 \otimes \Psi^2 \otimes \Psi^3 = \dots \\ \Phi^{-1} &= \sum \phi^1 \otimes \phi^2 \otimes \phi^3 = \sum \psi^1 \otimes \psi^2 \otimes \psi^3 = \dots \end{aligned}$$

Let $(A, m, u, \Delta, \varepsilon, \Phi, S)$ be a quasi-bialgebra with preantipode and consider the A -actions on $\text{End}(A) = \text{Hom}(A, A)$ defined by $(f \leftarrow a)(b) = f(ab)$ and $(a \rightarrow f)(b) = f(ba)$ for all $a, b \in A$ and for all $f \in \text{End}(A)$. Define the elements

$$(54) \quad p := \sum \varphi^1 \otimes \varphi^2 (\varphi^3 \rightarrow S) \in A \otimes \text{End}(A),$$

$$(55) \quad q := \sum (S \leftarrow \varphi^1) \varphi^2 \otimes \varphi^3 \in \text{End}(A) \otimes A,$$

where $(x(y \rightarrow f))(a) = xf(ay)$ and $((f \leftarrow x)y)(a) = f(ax)y$ for all $a, x, y \in A$ and for all $f \in \text{End}(A)$. Let us introduce the following notation for shortness:

$$p := \sum p^1 \otimes p^2 \quad \text{and} \quad q := \sum q^1 \otimes q^2.$$

Lemma A.1. *In the foregoing notation we have that for every $a \in A$*

$$(56) \quad \begin{aligned} \sum p^1 \otimes p^2(a) &= \sum \varphi_1^1 \psi^1 \otimes \varphi_2^1 \psi^2 \Phi^1 S(a \varphi^2 \psi_1^3 \Phi^2) \varphi^3 \psi_2^3 \Phi^3, \\ \sum q^1(a) \otimes q^2 &= \sum \Phi^1 \varphi_1^1 \psi^1 S(\Phi^2 \varphi_2^1 \psi^2 a) \Phi^3 \varphi^2 \psi_1^3 \otimes \varphi^3 \psi_2^3. \end{aligned}$$

Moreover, the following relations hold for every $a, b \in A$

$$(57) \quad \sum p^1 a \otimes p^2(b) = \sum a_{11} p^1 \otimes a_{12} p^2(ba_2),$$

$$(58) \quad \sum q^1(a) \otimes bq^2 = \sum q^1(b_1 a) b_{21} \otimes q^2 b_{22}.$$

Proof. The reassociator Φ satisfies the dual relation to (4), i.e.

$$(1 \otimes \Phi) \cdot (A \otimes \Delta \otimes A)(\Phi) \cdot (\Phi \otimes 1) = (A \otimes A \otimes \Delta)(\Phi) \cdot (\Delta \otimes A \otimes A)(\Phi).$$

In particular, it satisfies

$$\sum \varphi_1^1 \psi^1 \otimes \varphi_2^1 \psi^2 \Phi^1 \otimes \varphi^2 \psi_1^3 \Phi^2 \otimes \varphi^3 \psi_2^3 \Phi^3 = \sum \varphi^1 \psi^1 \otimes \varphi^2 \psi_1^2 \otimes \varphi^3 \psi_2^2 \otimes \psi^3.$$

Applying $(A \otimes m)(A \otimes A \otimes m)(A \otimes A \otimes (S \leftarrow a) \otimes A)$ to both sides we get

$$\begin{aligned} &\sum \varphi_1^1 \psi^1 \otimes \varphi_2^1 \psi^2 \Phi^1 S(a \varphi^2 \psi_1^3 \Phi^2) \varphi^3 \psi_2^3 \Phi^3 \\ &= \sum \varphi^1 \psi^1 \otimes \varphi^2 \psi_1^2 S(a \varphi^3 \psi_2^2) \psi^3 \stackrel{(53)}{=} \sum \varphi^1 \otimes \varphi^2 S(a \varphi^3) = \sum p^1 \otimes p^2(a), \end{aligned}$$

which is the first identity in (56). The second one is proved analogously. Let us check that (57) holds as well ((58) is proved similarly). We compute

$$\begin{aligned} \sum p^1 a \otimes p^2(b) &\stackrel{(54)}{=} \sum \varphi^1 a \otimes \varphi^2 S(b\varphi^3) \stackrel{(53)}{=} \sum \varphi^1 a_1 \otimes \varphi^2 a_{21} S(b\varphi^3 a_{22}) \\ &\stackrel{(*)}{=} \sum a_{11} \varphi^1 \otimes a_{12} \varphi^2 S(ba_2 \varphi^3) = \sum a_{11} p^1 \otimes a_{12} p^2(ba_2), \end{aligned}$$

where in $(*)$ we used the quasi-coassociativity $\Phi \cdot (\Delta \otimes A) \Delta = (A \otimes \Delta) \Delta \cdot \Phi$. \square

Lemma A.2. *Let $(A, m, u, \Delta, \varepsilon, \Phi, S)$ be a quasi-bialgebra with preantipode and let p, q be defined as above. For all $a \in A$ we have that*

$$S(a) = \sum q^1(1) S(p^1 a q^2) p^2(1) = \sum S(\varphi^1) \varphi^2 S(\psi^1 a \varphi^3) \psi^2 S(\psi^3).$$

Proof. Keeping in mind that Φ^{-1} is counital, i.e. that it satisfies

$$(\varepsilon \otimes A \otimes A) (\Phi^{-1}) = 1 \otimes 1 = (A \otimes \varepsilon \otimes A) (\Phi^{-1}) = 1 \otimes 1 = (A \otimes A \otimes \varepsilon) (\Phi^{-1}),$$

we may compute directly

$$\begin{aligned} \sum S(\varphi^1) \varphi^2 S(\psi^1 a \varphi^3) \psi^2 S(\psi^3) &= \sum q^1(1) S(p^1 a q^2) p^2(1) \\ &\stackrel{(56)}{=} \sum \Phi^1 \varphi_1^1 \psi^1 S(\Phi^2 \varphi_2^1 \psi^2) \Phi^3 \varphi^2 \psi_1^3 S(\gamma_1^1 \phi^1 a \varphi^3 \psi_2^3) \gamma_2^1 \phi^2 \Psi^1 S(\gamma^2 \phi_1^3 \Psi^2) \gamma^3 \phi_2^3 \Psi^3 \\ &\stackrel{(53)}{=} \sum \Phi^1 \varphi_1^1 S(\Phi^2 \varphi_2^1) \Phi^3 \varphi^2 S(\phi^1 a \varphi^3) \phi^2 \Psi^1 S(\phi_1^3 \Psi^2) \phi_2^3 \Psi^3 \\ &\stackrel{(53)}{=} \sum \Phi^1 S(\Phi^2) \Phi^3 S(a) \Psi^1 S(\Psi^2) \Psi^3 = S(a). \end{aligned} \quad \square$$

Proposition A.3. *Let $(A, m, u, \Delta, \varepsilon, \Phi, S)$ be a quasi-bialgebra with preantipode. For all $a, b \in A$ we have*

$$(59) \quad S(ab) = \sum S(\varphi^1 b) \varphi^2 S(\psi^1 \varphi^3) \psi^2 S(a \psi^3).$$

Proof. We know from Lemma A.2 that $S(a) = \sum q^1(1) S(p^1 a q^2) p^2(1)$. Relation (59) is proved directly by applying it to $S(ab)$:

$$\begin{aligned} S(ab) &= \sum q^1(1) S(p^1 a b q^2) p^2(1) \stackrel{(57)}{=} \sum q^1(1) S(a_{11} p^1 b q^2) a_{12} p^2(a_2) \\ &\stackrel{(53)}{=} \sum q^1(1) S(p^1 b q^2) p^2(a) \stackrel{(58)}{=} \sum q^1(b_1) b_{21} S(p^1 q^2 b_{22}) p^2(a) \\ &\stackrel{(53)}{=} \sum q^1(b) S(p^1 q^2) p^2(a) = \sum S(\varphi^1 b) \varphi^2 S(\psi^1 \varphi^3) \psi^2 S(a \psi^3). \end{aligned} \quad \square$$

Formula (59) can be viewed as an anti-multiplicativity of the preantipode.

APPENDIX B. FROM PREANTIPODES TO RIGIDITY

We want to provide an explicit construction for the right dual objects of finite-dimensional comodules over a coquasi-bialgebra with preantipode. As a byproduct, we will obtain a direct proof of the rigidity of the category of finite-dimensional corepresentations of a coquasi-bialgebra with preantipode.

Let H be a coquasi-bialgebra. Recall from [Sc4, Lemma 2.2] that ${}^H\mathfrak{M}_H^H$ is a monoidal category as follows. The tensor product of $M, N \in {}^H\mathfrak{M}_H^H$ is their cotensor product $M \square_H N$ with structures

$$\rho_{M \square_H N}^l(m \square_H n) = \sum m_{-1} \otimes (m_0 \square_H n),$$

$$\begin{aligned}\rho_{M \square_H N}^r(m \square_H n) &= \sum (m \square_H n_0) \otimes n_1, \\ (m \square_H n) h &= \sum m h_1 \square_H n h_2.\end{aligned}$$

The unit is H and the underlying functor ${}^H\mathfrak{M}_H^H \rightarrow {}^H\mathfrak{M}^H$ is a strict monoidal functor with the monoidal structure on the target given by cotensor product.

Lemma B.1 ([Sc4, Lemma 2.5]). *Let H be a coquasi-bialgebra, V be an object in ${}^H\mathfrak{M}_f$ and $F : {}^H\mathfrak{M} \rightarrow {}^H\mathfrak{M}_H^H$ be the functor introduced in §1.2. Then a right dual object to $M := F(V) = V \otimes H$ in $({}^H\mathfrak{M}_H^H, \square_H, H)$ is given by $(M^*, \text{EV}_M, \text{DB}_M)$ where $M^* = T(V^*) = V^* \otimes H$ with structures*

$$(60) \quad \begin{aligned}\rho_{M^*}^l(f \otimes h) &= \sum h_1 \otimes (f \otimes h_2), \quad \rho_{M^*}^r(f \otimes h) = \sum (f_0 \otimes h_1) \otimes f_1 h_2, \\ (f \otimes h) l &= \sum f_0 \otimes h_1 l_1 \omega(f_1 \otimes h_2 \otimes l_2),\end{aligned}$$

for all $f \in V^*$ and $h, l \in H$ (see (3) for the definition of $\sum f_0 \otimes f_1$). Furthermore, for all $v \in V$, $f \in V^*$ and $h, l \in H$ the maps EV_M and DB_M are given by

$$(61) \quad \text{EV}_M((v \otimes h) \square_H (f \otimes l)) = \sum v_{-1} \varepsilon(h) l f(v_0),$$

$$(62) \quad \text{DB}_M(h) = \sum_{i=1}^{d_V} (v^i \otimes h_1) \square_H (v_i \otimes h_2),$$

where $\sum_{i=1}^{d_V} v^i \otimes v_i$ is a dual basis for V as a vector space.

Corollary B.2. *Let H be a coquasi-bialgebra with preantipode S . Then the category ${}^H\mathfrak{M}_f$ is right rigid and for any V in ${}^H\mathfrak{M}_f$ a dual object is given by*

$$V^* = \left\{ \sum_t f^t \otimes h^t \in V^* \otimes H \mid \sum_t f_0^t \otimes h_1^t \otimes f_1^t h_2^t = \sum_t f^t \otimes h^t \otimes 1_H \right\}$$

with structure maps given for every $\sum_t f^t \otimes h^t \in V^*$ and for all $u \in V$ by $\rho_{V^*}(\sum_t f^t \otimes h^t) = \sum_t h_1^t \otimes (f^t \otimes h_2^t)$ and

$$\text{ev}_V \left(u \otimes \sum_t (f^t \otimes h^t) \right) = \sum_t f^t(u) \varepsilon(h^t), \quad \text{db}_V(1_{\mathbb{k}}) = \sum_{i=1}^{d_V} (v_0^i \otimes S(v_1^i)) \otimes v_i.$$

Proof. In view of [Sc4, Lemma 2.6] we know that the category ${}^H\mathfrak{M}_f$ is right rigid. Explicitly, since the functor F from §1.2 is an equivalence and a monoidal functor with quasi-inverse G , for any V in ${}^H\mathfrak{M}_f$ we have that $V \cong GF(V) = G(M)$, where $M = V \otimes H$, and hence we may choose as a right dual object $V^* := G(M^*) = (M^*)^{\text{co}H}$. It is finite-dimensional as well in view of relation (25). Recall (cf. [Sc4, Proposition 2.4]) that the monoidal structure on the functor F is given by the isomorphisms $\xi_0 := l_H^{-1} : H \rightarrow \mathbb{k} \otimes H$ and

$$\begin{aligned}\xi_{V,W} : (V \otimes H) \square_H (W \otimes H) &\rightarrow (V \otimes W) \otimes H, \\ \xi_{V,W} \left(\sum_i (u_i \otimes h_i) \square_H \sum_j (w_j \otimes h'_j) \right) &= \sum_{i,j} (u_i \otimes w_j) \otimes \varepsilon(h_i) h'_j.\end{aligned}$$

The inverse of $\xi_{V,W}$ is given for all $v \in V$, $w \in W$ and $h \in H$ by

$$\xi_{V,W}^{-1}((v \otimes w) \otimes h) = \sum (v \otimes w_{-1} h_1) \square_H (w_0 \otimes h_2).$$

Using the preantipode of H , let us give an explicit expression for $\text{ev}_V : V \otimes V^* \rightarrow \mathbb{k}$ and $\text{db}_V : \mathbb{k} \rightarrow V^* \otimes V$. We already know that monoidal functors preserve dual objects in the sense of Remark 2.2.6. In view of [AM, Proposition 3.84], the monoidal structure (ψ, ψ_0) on the functor G is given by

$$\psi_{N,P} := G(\epsilon_N \square_H \epsilon_P) G(\xi_{N,P}^{-1}) \eta_{G(N) \otimes G(P)}, \quad \psi_0 := G(\xi_0^{-1}) \eta_{\mathbb{k}}$$

for all N, P in ${}^H\mathfrak{M}_H^H$, where η and ϵ are the unit and the counit of the adjunction (F, G) given in (10). In view of [AP, Proposition 3.3 and Lemma 3.8], if for any $N \in {}^H\mathfrak{M}_H^H$ and for all $n \in N$ we set

$$(63) \quad \tau_N(n) := \sum \omega(n_{-1} \otimes S(n_1)_1 \otimes n_2) n_0 S(n_1)_2,$$

then (63) defines a map $\tau_N : N \rightarrow N^{coH}$ such that $\epsilon_N^{-1}(n) = \sum \tau_N(n_0) \otimes n_1$ for all $n \in N$. Therefore we have explicitly $\psi_0(k) = k1_H$ and

$$(64) \quad \psi_{N,P}(n \otimes p) = \sum np_{-1} \square_H p_0, \quad \psi_{N,P}^{-1} \left(\sum_i n_i \square_H p_i \right) = \sum_i \tau_N(n_i) \otimes p_i.$$

Set $1 = 1_H$. Since $\tau_{M^*}(f \otimes 1) = \sum f_0 \otimes S(f_1)$ for all $f \in V^*$, we may compute

$$\begin{aligned} \text{db}_V(1_{\mathbb{k}}) &= (V^* \otimes \eta_V^{-1})(\text{db}_{G(M)}(1_{\mathbb{k}})) = (V^* \otimes \eta_V^{-1})(\psi_{M^*,M}^{-1} G(\text{DB}_M) \psi_0(1_{\mathbb{k}})) \\ &\stackrel{(62)}{=} (V^* \otimes \eta_V^{-1}) \left(\psi_{M^*,M}^{-1} \left(\sum_i (v^i \otimes 1) \square_H (v_i \otimes 1) \right) \right) \\ &\stackrel{(64)}{=} (V^* \otimes \eta_V^{-1}) \left(\sum_i \tau_{M^*}(v^i \otimes 1) \otimes (v_i \otimes 1) \right) = \sum_i (v_0^i \otimes S(v_1^i)) \otimes v_i \end{aligned}$$

and

$$\begin{aligned} \text{ev}_V \left(u \otimes \sum_t (f^t \otimes h^t) \right) &= \text{ev}_{G(M)} \left((\eta_V \otimes G(M^*)) \left(u \otimes \sum_t (f^t \otimes h^t) \right) \right) \\ &\stackrel{(61)}{=} (\psi_0^{-1} G(\text{EV}_M) \psi_{M,M^*}) \left((u \otimes 1) \otimes \sum_t (f^t \otimes h^t) \right) \\ &= (\psi_0^{-1} G(\text{EV}_M)) \left(\sum_t (u \otimes h_1^t) \square_H (f^t \otimes h_2^t) \right) \\ &= \sum_t u_{-1} h^t f^t(u_0) = \sum_t f_0^t(u) \varepsilon(h_1^t) h_2^t f_1^t \stackrel{(*)}{=} \sum_t f^t(u) \varepsilon(h^t), \end{aligned}$$

where in $(*)$ we used that $\sum_t f^t \otimes h^t$ is a coinvariant element. \square

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DEPARTMENT OF MATHEMATICS "G. PEANO", UNIVERSITY OF TURIN, VIA CARLO ALBERTO 10, TORINO, I-10123, ITALY

URL: sites.google.com/site/paolosaracco

E-mail address: p.saracco@unito.it